

AN INVOLUTION OF PERIOD SEVENTEEN

By

JOHN WILLIS KENELLY, JR.

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

January, 1961

UNIVERSITY OF FLORIDA



3 1262 08552 2828

ACKNOWLEDGMENTS

The author expresses his sincere appreciation to Dr. W. R. Hutcherson, chairman of the supervisory committee, for his suggesting the problem and counseling before and during the preparation of this dissertation. Dr. Hutcherson has been a continuing source of inspiration, and his direction greatly contributed to making this study possible.

To Dr. J. E. Maxfield, Dr. T. O. Moore, Dr. W. P. Morse, and Dr. G. R. Bartlett the author expresses his appreciation for serving on the supervisory committee and for their assistance in editing this dissertation.

The typist, Mrs. Thomas Larrick, was also of great assistance in the final preparation.

TABLE OF CONTENTS

ACKNOWLEDGMENTS	Page ii
CHAPTER	
I. INTRODUCTION	1
II. A SURFACE ϕ OBTAINED FROM AN INVOLUTION OF PERIOD SEVENTEEN	3
1. The Image Surface \dagger	5
2. Branch Point O'_1	8
3. Branch Point O'_{10}	14
4. Branch Point O'_{11}	18
5. Multiplicities of Points O'_1 , O'_{10} , and O'_{11} for Surface ϕ	24
6. Summary	25
III. PROJECTIONS OF THE SURFACE ϕ	27
1. Surface ϕ_1	27
2. Surface \dagger_2	31
3. Surface \dagger_3	34
4. Surface ϕ_4	36
5. Surface \dagger_5	38
6. Surface ϕ_6	40
7. Surface ϕ_7	42
8. Summary	44
IV. A RATIONAL SURFACE F IN S_{11}	48
Conclusion	59

TABLE OF CONTENTS--(Continued)

	Page
APPENDIX	
I. A METHOD OF FINDING THE ORDER OF A QUINTIC TANGENT CONE	60
II. A METHOD OF INVESTIGATING A FOURTEENTH ORDER NEIGHBORHOOD	62
III. A METHOD OF DEMONSTRATING THE EXISTENCE OF $\Psi(x_1, x_2, \dots, x_{11})$	66
BIBLIOGRAPHY	69
BIOGRAPHICAL SKETCH	72

CHAPTER I

INTRODUCTION

In an extended complex plane with homogeneous coordinates the equations

$$x_1' : x_2' : x_3' = x_1 : Ex_2 : E^a x_3$$

define a plane cyclic homography of period p , where p is a prime number greater than two, E a p th primitive root of unity and a is an integer greater than unity and less than p . This homography generates an involution of period p .

Lucien Godeaux has been the world's leader in studying involutions. Since his paper in 1916 [5] where he used period three, he has published many papers on involutions. Many other authors have contributed to this field. Hutcherson studied involutions of period seven and eleven [14, 15], Childress studied some of period three, five, and thirteen [18, 19], Frank studied some of period eleven [3], and Gormsen studied some of period three, five, and seven [12].

This writer is investigating the mapping of an involution of period seventeen from a plane onto a surface in a space of ten dimensions (S_{10}). The three branch points of this surface ϕ require detailed study comprising Chapter II. In Chapter III certain projections of ϕ are investigated. A rational surface F , in S_{11} , is exhibited in Chapter IV. Points on this surface are in a one-to-one correspondence with points

on the original plane, whereas points of the surface ϕ are in a one-to-seventeen correspondence with points on the plane as well as with points on surface F.

The material in Chapter II was the subject of a joint paper given at the 1960 summer meeting of the American Mathematical Society [22]. The contents of Chapter IV were used in another joint paper given also at this meeting [21].

The reader is referred to the bibliography for introductory material to this area. One unfamiliar with the usage of terms, symbols, and techniques of this phase of Algebraic Geometry might not fully understand certain areas of this dissertation, e.g., first order neighborhoods [12]. Also homogeneous projective coordinates are used exclusively [13]. Since introductory material is plentiful and available it is usually omitted from most areas and references are mentioned instead.

As far as the author has been able to determine, most of this work is original.

CHAPTER II

A SURFACE ϕ OBTAINED FROM AN INVOLUTION OF PERIOD SEVENTEEN

1. The Image Surface ϕ

Consider the homography,

$$(H) \quad x'_1 : x'_2 : x'_3 = x_1 : Ex_2 : E^{15}x_3$$

where E is a primitive seventeenth root of unity. This homography generates an involution, I_{17} , of period seventeen. A group of I_{17} is composed of the following seventeen points,

$$\begin{aligned} & (x_1, x_2, x_3), (x_1, Ex_2, E^{15}x_3), (x_1, E^2x_2, E^{13}x_3), \\ & (x_1, E^3x_2, E^{11}x_3), (x_1, E^4x_2, E^9x_3), (x_1, E^5x_2, E^7x_3), \\ & (x_1, E^6x_2, E^5x_3), (x_1, E^7x_2, E^3x_3), (x_1, E^8x_2, Ex_3), \\ & (x_1, E^9x_2, E^{16}x_3), (x_1, E^{10}x_2, E^{14}x_3), (x_1, E^{11}x_2, E^{12}x_3), \\ & (x_1, E^{12}x_2, E^{10}x_3), (x_1, E^{13}x_2, E^8x_3), (x_1, E^{14}x_2, E^6x_3), \\ & (x_1, E^{15}x_2, E^4x_3), (x_1, E^{16}x_2, E^2x_3). \end{aligned}$$

Now consider the complete non-invariant linear system of order seventeen in the plane, i.e.,

$$\sum a_h x_1^i x_2^j x_3^k = 0$$

where $i + j + k = 17$ and $h = 1, 2, \dots, 171$ designates the different coefficients. In this general system there are seventeen systems of curves that are transformed into themselves by the homography H .

$$\begin{aligned}
 (1) \quad & a_1 x_1^{17} + a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 \\
 & + a_{59} x_1^7 x_2^9 x_3^3 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 \\
 & + a_{148} x_1 x_2^5 x_3^{11} + a_{170} x_2^{17} + a_{171} x_3^{17} = 0.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & g(x_1, x_2, x_3) = a_{17} x_1^{12} x_3^5 + a_{29} x_1^{10} x_2^7 + a_{42} x_1^9 x_2^2 x_3^6 \\
 & + a_{58} x_1^7 x_2^9 x_3 + a_{76} x_1^6 x_2^4 x_3^7 + a_{96} x_1^4 x_2^{11} x_3^2 + a_{119} x_1^3 x_2^6 x_3^8 \\
 & + a_{124} x_1^2 x_2 x_3^{14} + a_{143} x_1 x_2^{13} x_3^3 + a_{169} x_2^8 x_3^9 = 0.
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & a_{10} x_1^{14} x_2 x_3^2 + a_{28} x_1^{11} x_2^3 x_3^3 + a_{54} x_1^8 x_2^5 x_3^4 + a_{57} x_1^7 x_3^{10} \\
 & + a_{89} x_1^5 x_2^7 x_3^5 + a_{97} x_1^4 x_2^2 x_3^{11} + a_{106} x_1^3 x_2^{14} + a_{133} x_1^2 x_2^9 x_3^6 \\
 & + a_{146} x_1 x_2^4 x_3^{12} + a_{154} x_2^{16} x_3 = 0.
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & a_{11} x_1^{13} x_2^4 + a_{31} x_1^{10} x_2^6 x_3 + a_{40} x_1^9 x_2 x_3^7 + a_{60} x_1^7 x_2^8 x_3^2 \\
 & + a_{74} x_1^6 x_2^3 x_3^8 + a_{98} x_1^4 x_2^{10} x_3^3 + a_{117} x_1^3 x_2^5 x_3^9 + a_{122} x_1^2 x_3^{15} \\
 & + a_{145} x_1 x_2^{12} x_3^4 + a_{167} x_2^7 x_3^{10} = 0.
 \end{aligned}$$

$$\begin{aligned}
(5) \quad & a_8 x_1^{14} x_3^3 + a_{27} x_1^{11} x_2^2 x_3^4 + a_{55} x_1^8 x_2^4 x_3^5 + a_{67} x_1^6 x_2^{11} \\
& + a_{91} x_1^5 x_2^6 x_3^6 + a_{95} x_1^4 x_2 x_3^{12} + a_{108} x_1^3 x_2^{13} x_3 + a_{135} x_1^2 x_2^8 x_3^7 \\
& + a_{144} x_1 x_2^3 x_3^{13} + a_{156} x_2^{15} x_3^2 = 0.
\end{aligned}$$

$$\begin{aligned}
(6) \quad & a_2 x_1^{16} x_2 + a_{13} x_1^{13} x_2^3 x_3 + a_{33} x_1^{10} x_2^5 x_3^2 + a_{38} x_1^9 x_3^8 \\
& + a_{62} x_1^7 x_2^7 x_3^3 + a_{72} x_1^6 x_2^2 x_3^9 + a_{100} x_1^4 x_2^9 x_3^4 + a_{115} x_1^3 x_2^4 x_3^{10} \\
& + a_{147} x_1 x_2^{11} x_3^5 + a_{165} x_2^6 x_3^{11} = 0.
\end{aligned}$$

$$\begin{aligned}
(7) \quad & a_{25} x_1^{11} x_2 x_3^5 + a_{37} x_1^9 x_2^8 + a_{53} x_1^8 x_2^3 x_3^6 + a_{69} x_1^6 x_2^{10} x_3 \\
& + a_{90} x_1^5 x_2^5 x_3^7 + a_{93} x_1^4 x_3^{13} + a_{110} x_1^3 x_2^{12} x_3^2 + a_{136} x_1^2 x_2^7 x_3^8 \\
& + a_{142} x_1 x_2^2 x_3^{14} + a_{158} x_2^{14} x_3^3 = 0.
\end{aligned}$$

$$\begin{aligned}
(8) \quad & a_3 x_1^{16} x_3 + a_{15} x_1^{13} x_2^2 x_3^2 + a_{35} x_1^{10} x_2^4 x_3^3 + a_{64} x_1^7 x_2^6 x_3^4 \\
& + a_{70} x_1^6 x_2 x_3^{10} + a_{102} x_1^4 x_2^8 x_3^5 + a_{113} x_1^3 x_2^3 x_3^{11} + a_{121} x_1^2 x_2^{15} \\
& + a_{149} x_1 x_2^{10} x_3^6 + a_{163} x_2^5 x_3^{12} = 0.
\end{aligned}$$

$$\begin{aligned}
(9) \quad & a_{16} x_1^{12} x_2^5 + a_{23} x_1^{11} x_3^6 + a_{39} x_1^9 x_2^7 x_3 + a_{51} x_1^8 x_2^2 x_3^7 \\
& + a_{71} x_1^6 x_2^9 x_3^2 + a_{88} x_1^5 x_2^4 x_3^8 + a_{112} x_1^3 x_2^{11} x_3^3 + a_{134} x_1^2 x_2^6 x_3^9 \\
& + a_{140} x_1 x_2 x_3^{15} + a_{160} x_2^{13} x_3^4 = 0.
\end{aligned}$$

$$\begin{aligned}
 (10) \quad & a_{14} x_1^{13} x_2 x_3^3 + a_{36} x_1^{10} x_2^3 x_3^4 + a_{66} x_1^7 x_2^5 x_3^5 + a_{68} x_1^6 x_3^{11} \\
 & + a_{79} x_1^5 x_2^{12} + a_{104} x_1^4 x_2^7 x_3^6 + a_{111} x_1^3 x_2^2 x_3^{12} + a_{123} x_1^2 x_2^{14} x_3 \\
 & + a_{151} x_1 x_2^9 x_3^7 + a_{161} x_2^4 x_3^{13} = 0.
 \end{aligned}$$

$$\begin{aligned}
 (11) \quad & a_4 x_1^{15} x_2^2 + a_{18} x_1^{12} x_2^4 x_3 + a_{41} x_1^9 x_2^6 x_3^2 + a_{49} x_1^8 x_2 x_3^8 \\
 & + a_{73} x_1^6 x_2^8 x_3^3 + a_{86} x_1^5 x_2^3 x_3^9 + a_{114} x_1^3 x_2^{10} x_3^4 + a_{132} x_1^2 x_2^5 x_3^{10} \\
 & + a_{138} x_1 x_3^{16} + a_{162} x_2^{12} x_3^5 = 0.
 \end{aligned}$$

$$\begin{aligned}
 (12) \quad & a_{12} x_1^{13} x_3^4 + a_{34} x_1^{10} x_2^2 x_3^5 + a_{46} x_1^8 x_2^9 + a_{65} x_1^7 x_2^4 x_3^6 \\
 & + a_{81} x_1^5 x_2^{11} x_3 + a_{105} x_1^4 x_2^6 x_3^7 + a_{109} x_1^3 x_2 x_3^{13} + a_{125} x_1^2 x_2^{13} x_3^2 \\
 & + a_{153} x_1 x_2^8 x_3^8 + a_{159} x_2^3 x_3^{14} = 0.
 \end{aligned}$$

$$\begin{aligned}
 (13) \quad & a_6 x_1^{15} x_2 x_3 + a_{20} x_1^{12} x_2^3 x_3^2 + a_{43} x_1^9 x_2^5 x_3^3 + a_{47} x_1^8 x_3^9 \\
 & + a_{75} x_1^6 x_2^7 x_3^4 + a_{84} x_1^5 x_2^2 x_3^{10} + a_{116} x_1^3 x_2^9 x_3^5 + a_{130} x_1^2 x_2^4 x_3^{11} \\
 & + a_{137} x_1 x_2^{16} + a_{164} x_2^{11} x_3^6 = 0.
 \end{aligned}$$

$$\begin{aligned}
 (14) \quad & a_{22} x_1^{11} x_2^6 + a_{32} x_1^{10} x_2 x_3^6 + a_{48} x_1^8 x_2^8 x_3 + a_{63} x_1^7 x_2^3 x_3^7 \\
 & + a_{83} x_1^5 x_2^{10} x_3^2 + a_{103} x_1^4 x_2^5 x_3^8 + a_{107} x_1^3 x_3^{14} + a_{127} x_1^2 x_2^{12} x_3^3 \\
 & + a_{152} x_1 x_2^7 x_3^9 + a_{157} x_2^2 x_3^{15} = 0.
 \end{aligned}$$

$$\begin{aligned}
 (15) \quad & a_5 x_1^{15} x_3^2 + a_{21} x_1^{12} x_2^2 x_3^3 + a_{45} x_1^9 x_2^4 x_3^4 + a_{77} x_1^6 x_2^6 x_3^5 \\
 & + a_{82} x_1^5 x_2 x_3^{11} + a_{92} x_1^4 x_2^{13} + a_{118} x_1^3 x_2^8 x_3^6 + a_{128} x_1^2 x_2^3 x_3^{12} \\
 & + a_{139} x_1 x_2^{15} x_3 + a_{166} x_2^{10} x_3^7 = 0.
 \end{aligned}$$

$$\begin{aligned}
 (16) \quad & a_7 x_1^{14} x_2^3 + a_{24} x_1^{11} x_2^5 x_3 + a_{30} x_1^{10} x_3^7 + a_{50} x_1^8 x_2^7 x_3^2 \\
 & + a_{61} x_1^7 x_2^2 x_3^8 + a_{85} x_1^5 x_2^9 x_3^3 + a_{101} x_1^4 x_2^4 x_3^9 + a_{129} x_1^2 x_2^{11} x_3^4 \\
 & + a_{150} x_1 x_2^6 x_3^{10} + a_{155} x_2 x_3^{16} = 0.
 \end{aligned}$$

$$\begin{aligned}
 (17) \quad & a_{19} x_1^{12} x_2 x_3^4 + a_{44} x_1^9 x_2^3 x_3^5 + a_{56} x_1^7 x_2^{10} + a_{78} x_1^6 x_2^5 x_3^6 \\
 & + a_{80} x_1^5 x_3^{12} + a_{94} x_1^4 x_2^{12} x_3 + a_{120} x_1^3 x_2^7 x_3^7 + a_{126} x_1^2 x_2^2 x_3^{13} \\
 & + a_{141} x_1 x_2^{14} x_3^2 + a_{168} x_2^9 x_3^8 = 0.
 \end{aligned}$$

Now relate the curves of system (1) projectively to the hyperplanes of S_{10} by taking the projective transformation

$$\begin{aligned}
 (T) \quad & \frac{x_1}{x_1^{17}} = \frac{x_2}{x_1^{14} x_2^2 x_3} = \frac{x_3}{x_1^{11} x_2^4 x_3^2} = \frac{x_4}{x_1^8 x_2^6 x_3^3} = \frac{x_5}{x_1^7 x_2 x_3^9} \\
 & = \frac{x_6}{x_1^5 x_2^8 x_3^4} = \frac{x_7}{x_1^4 x_2^3 x_3^{10}} = \frac{x_8}{x_1^2 x_2^{10} x_3^5} = \frac{x_9}{x_1 x_2^5 x_3^{11}} \\
 & = \frac{x_{10}}{x_2^{17}} = \frac{x_{11}}{x_3^{17}}.
 \end{aligned}$$

Eliminate x_1 , x_2 , and x_3 in T to obtain a surface ϕ in S_{10} which has for its equations

$$(\phi) \quad \left\| \begin{array}{cccccccc} x_{10}x_1 & x_8x_5 & x_6 & x_4 & x_9x_5 & x_3 & x_7 & x_2x_5 & x_2^2 \\ x_4x_6 & x_7x_9 & x_9 & x_7 & x_7x_{11} & x_5 & x_{11} & x_1x_{11} & x_1x_5 \end{array} \right\| = 0.$$

This surface is the image of I_{17} , i.e., a set of I_{17} corresponds to a single point of the surface ϕ . Now investigate the singularities of the surface ϕ at the images of the fixed points of I_{17} . The images of O_1 (1, 0, 0), O_2 (0, 1, 0), and O_3 (0, 0, 1) are O'_1 (1,0,0,0,0,0,0,0,0,0,0), O'_{10} (0,0,0,0,0,0,0,0,0,1,0), and O'_{11} (0,0,0,0,0,0,0,0,0,0,1) respectively.

2. Branch Point O'_1

This investigation is based on a technique that finds the projective images of successive neighborhoods of the vertices of the triangle of reference in the plane. This definition of neighborhood is based on the existence of a quadratic transformation which relates two planes with homogeneous coordinates in such a manner that a reference triangle vertex in the z coordinate plane is mapped onto its corresponding reference triangle vertex in the x coordinate plane, but the x coordinate vertex is mapped to the meaningless point (0, 0, 0) in the z plane. For example, O_1 (1, 0, 0) in the z coordinate plane is mapped onto O_1 (1, 0, 0) in the x coordinate plane,

and $O_1 (1, 0, 0)$ in the x plane is mapped onto $(0, 0, 0)$ in the z coordinate plane. Then the z plane image of the point $S(1 + \lambda\alpha, \lambda\beta, \lambda\gamma)$ from the x plane, will be the image of the first order neighborhood of O_1 if the limit is taken as λ tends to zero. For a more detailed discussion of neighborhoods the reader is referred to Morelock [23].

Note that members of the family (1) do not in general go through the vertices of the reference triangle. But if the restriction $a_1 = 0$ is added, a new family (18) will go through.

$$(18) \quad a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 + a_{59} x_1^7 x_2 x_3^9 \\ + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 + a_{148} x_1 x_2^5 x_3^{11} \\ + a_{170} x_2^{17} + a_{171} x_3^{17} = 0.$$

The quadratic transformation R and its inverse will relate $P_1 (1, 0, 0)$ in the z coordinate plane to $O_1 (1, 0, 0)$ in the x coordinate plane and $O_1 (1, 0, 0)$ to the meaningless point $(0, 0, 0)$.

$$(R) \quad x_1 : x_2 : x_3 = z_1^2 : z_1 z_2 : z_2 z_3$$

$$(R)^{-1} \quad z_1 : z_2 : z_3 = x_1 x_2 : x_2^2 : x_1 x_3$$

Apply the transformation R successively fourteen times to equation (18) and arrive at

$$(19) \quad z_1^{238} (a_{170} z_2 + a_9 z_3) + a_{171} z_2^{222} z_3^{17} + a_{59} z_1^{119} z_2^{111} z_3^9 \\ + a_{26} z_1^{221} z_2^{16} z_3^2 + a_{99} z_1^{102} z_2^{127} z_3^{10} + a_{52} z_1^{204} z_2^{32} z_3^3 \\ + a_{87} z_1^{187} z_2^{48} z_3^4 + a_{148} z_1^{85} z_2^{143} z_3^{11} + a_{133} z_1^{170} z_2^{64} z_3^5 = 0.$$

This shows that the point ($z_2 = z_3 = 0$), corresponds to the point in the fourteenth order neighborhood of 0_1 in the direction of $x_3 = 0$, i.e., $0_{1222222222222222} = 0_{12}(14)$ [3, p. 32].

Now apply the transformation, R , fourteen successive times to the transformation

$$\begin{aligned}
 (20) \quad & \frac{x_2}{x_1^{14} x_2^2 x_3} = \frac{x_3}{x_1^{11} x_2^4 x_3^2} = \frac{x_4}{x_1^8 x_2^6 x_3^3} = \frac{x_5}{x_1^7 x_2 x_3^9} \\
 & = \frac{x_6}{x_1^5 x_2^8 x_3^4} = \frac{x_7}{x_1^4 x_2^3 x_3^{10}} = \frac{x_8}{x_1^2 x_2^{10} x_3^5} = \frac{x_9}{x_1 x_2^5 x_3^{11}} \\
 & = \frac{x_{10}}{x_2^{17}} = \frac{x_{11}}{x_3^{17}} .
 \end{aligned}$$

The simplified result is

$$\begin{aligned}
 (21) \quad & \frac{x_2}{z_1^{238} z_3} = \frac{x_3}{z_1^{221} z_2^{16} z_3^2} = \frac{x_4}{z_1^{204} z_2^{32} z_3^3} = \frac{x_5}{z_1^{119} z_2^{111} z_3^9} \\
 & = \frac{x_6}{z_1^{187} z_2^{48} z_3^4} = \frac{x_7}{z_1^{102} z_2^{127} z_3^{10}} = \frac{x_8}{z_1^{170} z_2^{64} z_3^5} \\
 & = \frac{x_9}{z_1^{85} z_2^{143} z_3^{11}} = \frac{x_{10}}{z_1^{238} z_2} = \frac{x_{11}}{z_2^{222} z_3^{17}} .
 \end{aligned}$$

A substitution of $z_3 = k z_2$ will allow an all directional approach to the point ($z_2 = z_3 = 0$). This substitution in (21), after simplification, gives

$$\begin{aligned}
 (22) \quad \frac{X_2}{k z_1^{238}} &= \frac{X_3}{k^2 z_1^{221} z_2^{17}} = \frac{X_4}{k^3 z_1^{204} z_2^{34}} = \frac{X_5}{k^9 z_1^{119} z_2^{119}} \\
 &= \frac{X_6}{k^4 z_1^{187} z_2^{51}} = \frac{X_7}{k^{10} z_1^{102} z_2^{136}} = \frac{X_8}{k^5 z_1^{170} z_2^{68}} \\
 &= \frac{X_9}{k^{11} z_1^{85} z_2^{153}} = \frac{X_{10}}{z_1^{238}} = \frac{X_{11}}{k^{17} z_2^{238}}.
 \end{aligned}$$

As z_2 tends to the limit zero, the above gives the equation of a plane tangent to ϕ at O_1' . It is

$$(23) \quad \begin{cases} X_2 = k X_{10} \\ X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = X_9 = X_{11} = 0. \end{cases}$$

Now examine a different quadratic transformation and its inverse

$$(S) \quad x_1 : x_2 : x_3 = z_1^2 : z_2 z_3 : z_1 z_3$$

$$(S)^{-1} \quad z_1 : z_2 : z_3 = x_1 x_3 : x_1 x_2 : x_3^2.$$

Apply it seven times to the curve (18), and the simplified result is

$$\begin{aligned}
 (24) \quad & z_1^{119} (a_{171} z_3^2 + a_{59} z_2 z_3 + a_9 z_2^2) + a_{170} z_2^{17} z_3^{104} \\
 & + a_{26} z_1^{102} z_2^4 z_3^{15} + a_{99} z_1^{102} z_2^3 z_3^{16} + a_{52} z_1^{85} z_2^6 z_3^{30} \\
 & + a_{87} z_1^{68} z_2^8 z_3^{45} + a_{148} z_1^{85} z_2^5 z_3^{31} + a_{133} z_1^{51} z_2^{10} z_3^{60} = 0.
 \end{aligned}$$

This shows that in the seventh order neighborhood of O_1 , the point on the curve (18) along the direction $x_2 = 0$ corresponds to the double point ($z_2 = z_3 = 0$).

Repeated application of the transformation S to the transformation (20), yields

$$\begin{aligned}
 (25) \quad & \frac{X_2}{z_1^{119} z_2^2} = \frac{X_3}{z_1^{102} z_2^4 z_3^{15}} = \frac{X_4}{z_1^{85} z_2^6 z_3^{30}} = \frac{X_5}{z_1^{119} z_2 z_3} \\
 & = \frac{X_6}{z_1^{68} z_2^8 z_3^{45}} = \frac{X_7}{z_1^{102} z_2^3 z_3^{16}} = \frac{X_8}{z_1^{51} z_2^{10} z_3^{60}} = \frac{X_9}{z_1^{85} z_2^5 z_3^{31}} \\
 & = \frac{X_{10}}{z_2^{17} z_3^{104}} = \frac{X_{11}}{z_1^{119} z_3^2}.
 \end{aligned}$$

Substitute $z_3 = k z_2$ to allow for an all directional approach to the point ($z_2 = z_3 = 0$). This simplifies to give

$$\begin{aligned}
 (26) \quad \frac{X_2}{z_1^{119}} &= \frac{X_3}{k^{15} z_1^{102} z_2^{17}} = \frac{X_4}{k^{30} z_1^{85} z_2^{34}} = \frac{X_5}{k z_1^{119}} \\
 &= \frac{X_6}{k^{45} z_1^{68} z_2^{51}} = \frac{X_7}{k^{16} z_1^{102} z_2^{17}} = \frac{X_8}{k^{60} z_1^{51} z_2^{68}} \\
 &= \frac{X_9}{k^{31} z_1^{85} z_2^{34}} = \frac{X_{10}}{k^{104} z_2^{119}} = \frac{X_{11}}{k^2 z_1^{119}} .
 \end{aligned}$$

Now let z_2 approach the limit zero, and the equation of the other tangent element to ϕ at O'_1 is

$$(27) \quad \begin{cases} X_5^2 - X_{11} X_2 = 0 \\ X_3 = X_4 = X_6 = X_7 = X_8 = X_9 = X_{10} = 0. \end{cases}$$

This surface (27) when investigated is seen to be a quadric cone.[†] The reader is referred to Gormsen [12] for a different method for this investigation, i.e., Coble's method.

Hence, the following

Theorem 1: The tangent elements to the surface ϕ at the point O'_1 (1,0,0,0,0,0,0,0,0,0) are a plane (23) and a quadric cone (27).

[†]See Appendix I.

3. Branch Point O_{10}^I

The point O_2 (0, 1, 0) corresponds to the point O_{10}^I (0,0,0,0,0,0,0,0,0,1,0) in S_{10} by the transformation T. To study the tangent elements at this point, examine the system (∞^9) of curves passing through O_2 , i.e., system (1) with $a_{170} = 0$. We have

$$(28) \quad \begin{aligned} & a_1 x_1^{17} + a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 \\ & + a_{59} x_1^7 x_2 x_3^9 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 \\ & + a_{148} x_1 x_2^5 x_3^{11} + a_{171} x_3^{17} = 0. \end{aligned}$$

Apply the quadratic transformation

$$(U) \quad x_1 : x_2 : x_3 = z_1 z_2 : z_2^2 : z_1 z_3$$

$$(U)^{-1} \quad z_1 : z_2 : z_3 = x_1^2 : x_1 x_2 : x_2 x_3$$

twice in succession to the system (28). We get

$$(29) \quad \begin{aligned} & z_2^{34} (a_1 x_1^5 + a_9 z_1^4 z_3 + a_{26} x_1^3 x_3^2 + a_{52} z_1^2 z_3^3 + a_{87} z_1 z_3^4 \\ & + a_{133} z_3^5) + a_{171} z_1^{22} z_3^{17} + a_{59} z_1^{13} z_2^{17} z_3^9 + a_{99} z_1^{12} z_2^{17} z_3^{10} \\ & + a_{148} z_1^{11} z_2^{17} z_3^{11} = 0. \end{aligned}$$

This indicates that O_{211} , the second order neighborhood point in the $x_3 = 0$ direction, corresponds to the five tuple point ($z_1 = z_3 = 0$).

Now apply the transformation U twice to the transformation

$$\begin{aligned}
 (30) \quad \frac{X_1}{x_1^{17}} &= \frac{X_2}{x_1^{14} x_2^2 x_3} = \frac{X_3}{x_1^{11} x_2^4 x_3^2} = \frac{X_4}{x_1^8 x_2^6 x_3^3} \\
 &= \frac{X_5}{x_1^7 x_2^9 x_3^5} = \frac{X_6}{x_1^5 x_2^8 x_3^4} = \frac{X_7}{x_1^4 x_2^3 x_3^{10}} = \frac{X_8}{x_1^2 x_2^{10} x_3^5} \\
 &= \frac{X_9}{x_1^5 x_2^{11} x_3^{11}} = \frac{X_{11}}{x_3^{17}} .
 \end{aligned}$$

This gives

$$\begin{aligned}
 (31) \quad \frac{X_1}{z_1^5 z_2^{34}} &= \frac{X_2}{z_1^4 z_2^{34} z_3} = \frac{X_3}{z_1^3 z_2^{34} z_3^2} = \frac{X_4}{z_1^2 z_2^{34} z_3^3} \\
 &= \frac{X_5}{z_1^{13} z_2^{17} z_3^9} = \frac{X_6}{z_1 z_2^{34} z_3^4} = \frac{X_7}{z_1^{12} z_2^{17} z_3^{10}} = \frac{X_8}{z_2^{34} z_3^5} \\
 &= \frac{X_9}{z_1^{11} z_2^{17} z_3^{11}} = \frac{X_{11}}{z_1^{22} z_3^{17}} .
 \end{aligned}$$

Let $z_1 = k z_3$, to allow an approach from all directions to the image point ($z_1 = z_3 = 0$),

$$\begin{aligned}
 (32) \quad \frac{x_1}{k^5 z_2^{34}} &= \frac{x_2}{k^4 z_2^{34}} = \frac{x_3}{k^3 z_2^{34}} = \frac{x_4}{k^2 z_2^{34}} = \frac{x_5}{k^{13} z_2^{17} z_3^{17}} \\
 &= \frac{x_6}{k z_2^{34}} = \frac{x_7}{k^{12} z_2^{17} z_3^{17}} = \frac{x_8}{z_2^{34}} = \frac{x_9}{k^{11} z_2^{17} z_3^{17}} = \frac{x_{11}}{k^{22} z_3^{34}} .
 \end{aligned}$$

As z_3 approaches the limit zero in (32), the equation of a tangent element is arrived at

$$(33) \quad \left\{ \begin{array}{l} \left\| \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_6 \\ x_2 & x_3 & x_4 & x_6 & x_8 \end{array} \right\| = 0 \\ x_5 = x_7 = x_9 = x_{11} = 0. \end{array} \right.$$

This tangent surface is verified to be a quintic cone when investigated.[†]

To study the neighborhood points along the direction $x_1 = 0$, examine (28) with the quadratic transformation

$$(V) \quad x_1 : x_2 : x_3 = z_1 z_3 : z_2^2 : z_2 z_3$$

$$(V)^{-1} \quad z_1 : z_2 : z_3 = x_1 x_2 : x_2 x_3 : x_3^2.$$

Apply the transformation V five successive times to equation (28); this gives

[†]See Appendix I.

$$\begin{aligned}
 (34) \quad & z_2^{85} (a_{171} z_3^2 + a_{148} z_1 z_3 + a_{133} z_1^2) + a_1 z_1^{17} z_3^{70} \\
 & + a_{59} z_1^7 z_2^{51} z_3^{29} + a_9 z_1^{14} z_2^{17} z_3^{56} + a_{26} z_1^{11} z_2^{34} z_3^{42} \\
 & + a_{99} z_1^4 z_2^{68} z_3^{15} + a_{52} z_1^8 z_2^{51} z_3^{28} + a_{87} z_1^5 z_2^{68} z_3^{14} = 0.
 \end{aligned}$$

This indicates that the fifth order neighborhood point ($0_{23(5)}$) in the direction $x_1 = 0$, corresponds to the double point ($z_1 = z_3 = 0$).

Now apply the transformation V repeatedly five times to the transformation (30), and get

$$\begin{aligned}
 (35) \quad & \frac{x_1}{z_1^{17} z_3^{70}} = \frac{x_2}{z_1^{14} z_2^{17} z_3^{56}} = \frac{x_3}{z_1^{11} z_2^{34} z_3^{42}} = \frac{x_4}{z_1^8 z_2^{51} z_3^{28}} \\
 & = \frac{x_5}{z_1^7 z_2^{51} z_3^{29}} = \frac{x_6}{z_1^5 z_2^{68} z_3^{14}} = \frac{x_7}{z_1^4 z_2^{68} z_3^{14}} = \frac{x_8}{z_1^2 z_2^{85}} \\
 & = \frac{x_9}{z_1 z_2^{85} z_3} = \frac{x_{11}}{z_2^{85} z_3^2}.
 \end{aligned}$$

To allow for an approach from any direction to the point ($z_1 = z_3 = 0$), let $z_3 = k z_1$, and obtain

$$\begin{aligned}
 (36) \quad & \frac{X_1}{k^{70} z_1^{85}} = \frac{X_2}{k^{56} z_1^{68} z_2^{17}} = \frac{X_3}{k^{42} z_1^{51} z_2^{34}} = \frac{X_4}{k^{28} z_1^{34} z_2^{51}} \\
 & = \frac{X_5}{k^{29} z_1^{34} z_2^{51}} = \frac{X_6}{k^{14} z_1^{17} z_2^{68}} = \frac{X_7}{k^{15} z_1^{17} z_2^{68}} = \frac{X_8}{z_2^{85}} \\
 & = \frac{X_9}{k z_2^{85}} = \frac{X_{11}}{k^2 z_2^{85}} .
 \end{aligned}$$

As z_1 tends to the limit zero in (36) the other tangent element is given as

$$(37) \quad \begin{cases} X_9^2 - X_8 X_{11} = 0 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = 0. \end{cases}$$

This is seen to be a quadric cone when the order is determined.

Thus,

Theorem 2: The tangent elements to the surface ϕ at the point $O_{10}^0 (0,0,0,0,0,0,0,0,0,1,0)$ are a quintic cone (33) and a quadric cone (37).

4. Branch Point O_{11}^1

The point $O_3 (0, 0, 1)$ on the plane corresponds under the transformation T to the point $O_{11}^1 (0,0,0,0,0,0,0,0,0,1)$. To study O_{11}^1 investigate the system of curves that are members of (1) and have the restriction $a_{171} = 0$. They are

$$\begin{aligned}
 (38) \quad & a_1 x_1^{17} + a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 \\
 & + a_{59} x_1^7 x_2 x_3^9 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 \\
 & + a_{148} x_1 x_2^5 x_3^{11} + a_{170} x_2^{17} = 0.
 \end{aligned}$$

Apply the transformation

$$(M) \quad x_1 : x_2 : x_3 = z_1 z_2 : z_2 z_3 : z_3^2$$

$$(M)^{-1} \quad z_1 : z_2 : z_3 = x_1 x_3 : x_2^2 : x_2 x_3$$

eleven times in succession to (38). The result is

$$\begin{aligned}
 (39) \quad & z_3^{187} (a_{170} z_2 + a_{148} z_1) + a_1 z_1^{17} z_2^{171} + a_{59} z_1^7 z_2^{62} z_3^{119} \\
 & + a_9 z_1^{14} z_2^{140} z_3^{34} + a_{26} z_1^{11} z_2^{109} z_3^{68} + a_{99} z_1^4 z_2^{31} z_3^{153} \\
 & + a_{52} z_1^8 z_2^{78} z_3^{102} + a_{87} z_1^5 z_2^{47} z_3^{136} + a_{133} z_1^2 z_2^{16} z_3^{170} = 0.
 \end{aligned}$$

Hence the eleventh order neighborhood point $O_{32(11)}$, in the direction $x_1 = 0$, corresponds to the simple point $(z_1 = z_2 = 0)$.

Now apply the transformation M eleven times to the transformation

$$\begin{aligned}
 (40) \quad & \frac{x_1}{x_1^{17}} = \frac{x_2}{x_1^{14} x_2^2 x_3} = \frac{x_3}{x_1^{11} x_2^4 x_3^2} = \frac{x_4}{x_1^8 x_2^6 x_3^3} = \frac{x_5}{x_1^7 x_2 x_3^9} \\
 & = \frac{x_6}{x_1^5 x_2^8 x_3^4} = \frac{x_7}{x_1^4 x_2^3 x_3^{10}} = \frac{x_8}{x_1^7 x_2^{10} x_3^5} = \frac{x_9}{x_1 x_2^5 x_3^{11}} = \frac{x_{10}}{x_2^{17}}.
 \end{aligned}$$

The result is

$$\begin{aligned}
 (41) \quad & \frac{X_1}{z_1^{17} z_2^{171}} = \frac{X_2}{z_1^{14} z_2^{140} z_3^{34}} = \frac{X_3}{z_1^{11} z_2^{109} z_3^{68}} = \frac{X_4}{z_1^8 z_2^{78} z_3^{102}} \\
 & = \frac{X_5}{z_1^7 z_2^{62} z_3^{119}} = \frac{X_6}{z_1^5 z_2^{47} z_3^{136}} = \frac{X_7}{z_1^4 z_2^{31} z_3^{153}} = \frac{X_8}{z_1^2 z_2^{16} z_3^{170}} \\
 & = \frac{X_9}{z_1 z_3^{187}} = \frac{X_{10}}{z_2 z_3^{187}}.
 \end{aligned}$$

To allow for an all directional approach to the point

($z_1 = z_2 = 0$), let $z_2 = k z_1$, and get

$$\begin{aligned}
 (42) \quad & \frac{X_1}{k^{171} z_1^{187}} = \frac{X_2}{k^{140} z_1^{153} z_3^{34}} = \frac{X_3}{k^{109} z_1^{119} z_3^{68}} = \frac{X_4}{k^{78} z_1^{85} z_3^{102}} \\
 & = \frac{X_5}{k^{62} z_1^{68} z_3^{119}} = \frac{X_6}{k^{47} z_1^{51} z_3^{136}} = \frac{X_7}{k^{31} z_1^{34} z_3^{153}} \\
 & = \frac{X_8}{k^{16} z_1^{17} z_3^{170}} = \frac{X_9}{z_3^{187}} = \frac{X_{10}}{k z_3^{187}}.
 \end{aligned}$$

As z_1 approaches the limit zero, we arrive at the equation of a tangent plane,

$$(43) \quad \begin{cases} X_{10} = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0. \end{cases}$$

Now consider the quadratic transformation

$$(L) \quad x_1 : x_2 : x_3 = z_1 z_3 : z_1 z_2 : z_3^2$$

$$(L)^{-1} \quad z_1 : z_2 : z_3 = x_1^2 : x_2 x_3 : x_1 x_3.$$

Nine successive applications of L to equation (38) give

$$(44) \quad z_3^{153} (a_1 z_1 + a_{59} z_2) + a_{170} z_1^{137} z_2^{17} + a_9 z_1^{16} z_2^2 z_3^{136} \\ + a_{26} z_1^{31} z_2^4 z_3^{119} + a_{99} z_1^{15} z_2^3 z_3^{136} + a_{52} z_1^{46} z_2^6 z_3^{102} \\ + a_{87} z_1^{61} z_2^8 z_3^{85} + a_{148} z_1^{30} z_2^5 z_3^{119} + a_{133} z_1^{76} z_2^{10} z_3^{68} = 0.$$

This indicates that the point in the ninth order neighborhood in the direction $x_2 = 0$, $0_{31}(9)$, corresponds to the simple point ($z_1 = z_2 = 0$).

The transformation (40) under nine successive applications of L gives

$$(45) \quad \frac{x_1}{z_1 z_3^{153}} = \frac{x_2'}{z_1^{16} z_2^2 z_3^{136}} = \frac{x_3}{z_1^{31} z_2^4 z_3^{119}} = \frac{x_4}{z_1^{46} z_2^6 z_3^{102}} \\ = \frac{x_5}{z_2 z_3^{153}} = \frac{x_6}{z_1^{61} z_2^8 z_3^{85}} = \frac{x_7}{z_1^{15} z_2^3 z_3^{136}} = \frac{x_8}{z_1^{76} z_2^{10} z_3^{68}} \\ = \frac{x_9}{z_1^{30} z_2^5 z_3^{119}} = \frac{x_{10}}{z_1^{137} z_2^{17}}.$$

Approach the point ($z_1 = z_2 = 0$) from all directions by substituting $z_1 = k z_2$,

$$\begin{aligned}
 (46) \quad & \frac{X_1}{k \begin{smallmatrix} 153 \\ z_3 \end{smallmatrix}} = \frac{X_2}{k \begin{smallmatrix} 16 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 17 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 136 \\ z_3 \end{smallmatrix}} = \frac{X_3}{k \begin{smallmatrix} 31 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 34 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 119 \\ z_3 \end{smallmatrix}} = \frac{X_4}{k \begin{smallmatrix} 46 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 51 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 102 \\ z_3 \end{smallmatrix}} \\
 & = \frac{X_5}{\begin{smallmatrix} 153 \\ z_3 \end{smallmatrix}} = \frac{X_6}{k \begin{smallmatrix} 61 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 68 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 85 \\ z_3 \end{smallmatrix}} = \frac{X_7}{k \begin{smallmatrix} 15 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 17 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 136 \\ z_3 \end{smallmatrix}} = \frac{X_8}{k \begin{smallmatrix} 76 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 85 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 68 \\ z_3 \end{smallmatrix}} \\
 & = \frac{X_9}{k \begin{smallmatrix} 30 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 34 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 119 \\ z_3 \end{smallmatrix}} = \frac{X_{10}}{k \begin{smallmatrix} 137 \\ z_2 \end{smallmatrix} \begin{smallmatrix} 153 \\ z_2 \end{smallmatrix}} .
 \end{aligned}$$

As z_2 tends to the limit zero, equations (46) give another tangent plane,

$$(47) \quad \begin{cases} X_1 = k X_5 \\ X_2 = X_3 = X_4 = X_6 = X_7 = X_8 = X_9 = X_{10} = 0. \end{cases}$$

Note that when the transformation L is applied to equation (38) and then the transformation M is applied, the result is

$$\begin{aligned}
 (48) \quad & z_3^{34} (a_{59} z_1^2 + a_{99} z_1 z_2 + a_{148} z_2^2) + a_1 z_1^{11} z_2^8 z_3^{17} \\
 & + a_{170} z_1^{11} z_2^{25} + a_9 z_1^{10} z_2^9 z_3^{17} + a_{26} z_1^9 z_2^{10} z_3^{17} \\
 & + a_{52} z_1^8 z_2^{11} z_3^{17} + a_{87} z_1^7 z_2^{12} z_3^{17} + a_{133} z_1^6 z_2^{13} z_3^{17} = 0.
 \end{aligned}$$

This indicates that the second order neighborhood point, O_{312} , corresponds to the double point ($z_1 = z_2 = 0$).

The application of L followed by M to the transformation (40) gives

$$\begin{aligned}
 (49) \quad \frac{X_1}{z_1^{11} z_2^8 z_3^{17}} &= \frac{X_2}{z_1^{10} z_2^9 z_3^{17}} = \frac{X_3}{z_1^9 z_2^{10} z_3^{17}} = \frac{X_4}{z_1^8 z_2^{11} z_3^{17}} \\
 &= \frac{X_5}{z_1^2 z_2^{34}} = \frac{X_6}{z_1^7 z_2^{12} z_3^{17}} = \frac{X_7}{z_1 z_2 z_3^{34}} = \frac{X_8}{z_1^6 z_2^{13} z_3^{17}} \\
 &= \frac{X_9}{z_2^2 z_3^{34}} = \frac{X_{10}}{z_1^{11} z_2^{25}} .
 \end{aligned}$$

Now approach the point ($z_1 = z_2 = 0$) from all directions by making the substitution $z_2 = k z_1$,

$$\begin{aligned}
 (50) \quad \frac{X_1}{k^8 z_1^{17} z_3^{17}} &= \frac{X_2}{k^9 z_1^{17} z_3^{17}} = \frac{X_3}{k^{10} z_1^{17} z_3^{17}} = \frac{X_4}{k^{11} z_1^{17} z_3^{17}} \\
 &= \frac{X_5}{z_3^{34}} = \frac{X_6}{k^{12} z_1^{17} z_3^{17}} = \frac{X_7}{k z_3^{34}} = \frac{X_8}{k^{13} z_1^{17} z_3^{17}} = \frac{X_9}{k^2 z_3^{34}} \\
 &= \frac{X_{10}}{k^{25} z_1^{34}} .
 \end{aligned}$$

As z_1 approaches the limit zero, a third tangent element is arrived at. It is

$$(51) \quad \begin{cases} X_7^2 - X_5 X_9 = 0 \\ X_1 = X_2 = X_3 = X_4 = X_6 = X_8 = X_{10} = 0. \end{cases}$$

This surface is demonstrated to be a quadric cone, when investigated.

Thus, the following

Theorem 3: The tangent elements to ϕ at the point

O'_{11} (0,0,0,0,0,0,0,0,0,0,1) are two planes (43), (47), and a quadric cone (51).

5. Multiplicities of Points O'_1 , O'_{10} , and O'_{11} for Surface ϕ

The surface ϕ is of order 17. Two members of the family (18) intersect at O_1 , $14 \cdot 1^2 + 7 \cdot 2^2 + 1 \cdot 3^2$ or 51 times [1, p. 30]. Thus, the system is of degree 289 - 51 or 238.[†] Since the curves (18) are related projectively to the hyperplanes of S_{10} , the multiplicity of the point O'_1 on ϕ is 51/17 or 3.

Two members of the family (28) intersect at O_2 in $1 \cdot 7^2 + 5 \cdot 2^2 + 2 \cdot 5^2$ or 119 fixed points. Since the system intersects in 289 - 119 variable points it is of degree 170. Also, the multiplicity of O'_{10} on ϕ is 119/17 or 7.

[†]Degree is used in the same sense as Godeaux [5].

The tangent elements at O'_1 constitute a plane and a quadric cone and the tangent elements at O'_{10} are a quintic cone and a quadric cone. The branch point O'_{11} is more interesting in that it has two tangent planes and a quadric cone.

CHAPTER III

PROJECTIONS OF THE SURFACE ϕ

1. Surface ϕ_1

The surface ϕ projects from the point

O'_{10} (0,0,0,0,0,0,0,0,0,1,0) to the surface ϕ_1 in the space $X_{10} = 0$.

The equations for the surface ϕ_1 are

$$(\phi_1) \quad \left\{ \begin{array}{l} \left| \begin{array}{cccccccc} X_8 X_5 & X_6 & X_4 & X_9 X_5 & X_3 & X_7 & X_2 X_5 & X_2^2 \\ X_7 X_9 & X_9 & X_7 & X_7 X_{11} & X_5 & X_{11} & X_1 X_{11} & X_1 X_5 \end{array} \right| = 0, \\ X_{10} = 0. \end{array} \right.$$

Two members of the family (28) intersect in $1.7^2 + 5.2^2 + 2.5^2$ or 119 fixed points. Thus the order of ϕ_1 is $(289 - 119)/17$ or 10 (cf., Chapter II, Figure 1).

Now examine the family of curves (which pass through the point O_2 (0, 1, 0))

$$(52) \quad \begin{aligned} & a_{87} x_1^5 x_2^8 x_3^4 + a_{52} x_1^8 x_2^6 x_3^3 + a_{148} x_1 x_2^5 x_3^{11} + a_{26} x_1^{11} x_2^4 x_3^2 \\ & + a_{99} x_1^4 x_2^3 x_3^{10} + a_9 x_1^{14} x_2^2 x_3 + a_{59} x_1^7 x_2 x_3^9 + a_1 x_1^{17} \\ & + a_{171} x_3^{17} = 0. \end{aligned}$$

Notice that the point O_2 is a nine tuple point.

Apply the quadratic transformation U twice to the family (52).

The result is

$$\begin{aligned}
 (53) \quad & z_2^{34} (a_1 z_1^4 + a_9 z_1^3 z_3 + a_{26} z_1^2 z_3^2 + a_{52} z_1 z_3^3 + a_{87} z_3^4) \\
 & + a_{148} z_1^{10} z_2^{17} z_3^{11} + a_{99} z_1^{11} z_2^{17} z_3^{10} + a_{59} z_1^{12} z_2^{17} z_3^9 \\
 & + a_{171} z_1^{21} z_3^{17} = 0.
 \end{aligned}$$

Now apply the quadratic transformation V five successive times to (52) to obtain

$$\begin{aligned}
 (54) \quad & z_2^{85} (a_{148} z_1 + a_{171} z_3) + a_{87} z_1^5 z_2^{68} z_3^{13} + a_{52} z_1^8 z_2^{51} z_3^{27} \\
 & + a_{26} z_1^{11} z_2^{34} z_3^{41} + a_{99} z_1^4 z_2^{68} z_3^{14} + a_9 z_1^{14} z_2^{17} z_3^{55} \\
 & + a_{59} z_1^7 z_2^{51} z_3^{28} + a_1 z_1^{17} z_3^{69} = 0.
 \end{aligned}$$

Also apply to (52) the quadratic transformation V, then U, and then V twice in succession. The result is

$$\begin{aligned}
 (55) \quad & z_2^{68} (a_{87} z_1 + a_{148} z_3) + a_{52} z_1^6 z_2^{51} z_3^{12} + a_{26} z_1^{11} z_2^{34} z_3^{24} \\
 & + a_{99} z_1^5 z_2^{47} z_3^{13} + a_9 z_1^{16} z_2^{17} z_3^{36} + a_{59} z_1^{10} z_2^{34} z_3^{25} \\
 & + a_1 z_1^{21} z_3^{48} + a_{171} z_1^4 z_2^{51} z_3^{14} = 0.
 \end{aligned}$$

The three previous results indicate that the curves of system (52) have in common in the neighborhood of O_2 :

- (a) two successive four tuple points O_{21} and O_{211} ;
- (b) a four tuple point O_{23} ;
- (c) four successive simple points O_{233} , O_{2333} , O_{23333} ,
and O_{233333} ;
- (d) three successive simple points O_{231} , O_{2313} , and
 O_{23133} .

Hence, two curves of system (52) intersect $9^2 + 3 \cdot 4^2 + 7 \cdot 1^2$ or 136 times at O_2 . Therefore, the system (52) has degree $289 - 136$ or 153. The sum of the multiplicity of O_{10}' for surface ϕ and the multiplicity of O_8' for ϕ_1 is $136/17$ or 8. But O_{10}' is multiple of order 7 for ϕ . Hence O_8' is multiple of order 1 for ϕ_1 .

Now in a manner similar to the material in Chapter II, apply the quadratic transformation U twice to the projectivity obtained from equation (52) and substitute $z_1 = k z_3$. As z_3 tends to the limit zero, the result is,

$$(56) \quad \left\{ \begin{array}{l} \left| \begin{array}{cccc} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_6 \end{array} \right| = 0, \\ X_5 = X_7 = X_9 = X_{10} = X_{11} = 0. \end{array} \right.$$

Hence certain points of ϕ_1 , infinitely near O_8' , situated on (56), correspond to the points infinitely near O_{211} . Note that this is a projection of (33) to the space $X_{10} = 0$.

Apply the quadratic transformation V five successive times to the projectivity and substitute $z_3 = k z_1$. As z_1 approaches the limit zero, one gets the equations

$$(57) \quad \begin{cases} X_{11} = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_{10} = 0. \end{cases}$$

Hence certain points on ϕ_1 , infinitely near O_8^1 , situated on (57), correspond to the points infinitely near O_{233333} . Note that this is the projection of (37) to the space $X_{10} = 0$.

Now apply to the projectivity the quadratic transformation V , then U , and then V two successive times. Substitute $z_3 = k z_1$ and take the limit as z_1 approaches zero, and obtain

$$(58) \quad \begin{cases} X_6 = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = X_{10} = X_{11} = 0. \end{cases}$$

Hence certain points of ϕ_1 , infinitely near O_8^1 , situated on (58), correspond to the points infinitely near O_{23133} .

Since (56) and (57) were projections of previous tangent elements, our new tangent element is the plane that projects (58) from O_8^1 . Hence, the following

Theorem 4: The surface ϕ_1 has a new tangent element

$$(59) \quad \begin{cases} X_6 = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = X_{10} = X_{11} = 0 \end{cases}$$

at the point O_8^1 .

2. Surface ϕ_2

Project the surface ϕ_1 from the point $O'_8 (0,0,0,0,0,0,0,1,0,0,0)$ into the surface ϕ_2 in the space $X_8 = 0$, getting

$$(\phi_2) \quad \left\{ \begin{array}{l} \left\| \begin{array}{cccccc} X_6 & X_4 & X_9 X_5 & X_3 & X_7 & X_2 X_5 & X_2^2 \\ X_9 & X_7 & X_7 X_{11} & X_5 & X_{11} & X_1 X_{11} & X_1 X_5 \end{array} \right\| = 0, \\ X_{10} = X_8 = 0. \end{array} \right.$$

Two members of the family (52) intersect in $1 \cdot 9^2 + 3 \cdot 4^2 + 7 \cdot 1^2$ or 136 fixed points. Thus the order of ϕ_2 is $(289 - 136)/17$ or 9.

Examine the family of curves which pass through $O_2 (0, 1, 0)$,

$$(60) \quad a_{52} x_1^3 x_2^6 x_3^3 + a_{148} x_1 x_2^5 x_3^{11} + a_{26} x_1^{11} x_2^4 x_3^2 + a_{99} x_1^4 x_2^3 x_3^{10} \\ + a_9 x_1^{14} x_2^2 x_3 + a_{59} x_1^7 x_2 x_3^9 + a_1 x_1^{17} + a_{171} x_3^{17} = 0.$$

Note that O_2 is an eleven tuple point.

Apply the quadratic transformation U twice in succession to obtain

$$(61) \quad z_2^{34} (a_1 z_1^3 + a_9 z_1^2 z_3 + a_{26} z_1 z_3^2 + a_{52} z_3^3) + a_{148} z_1^9 z_2^{17} z_3^{11} \\ + a_{99} z_1^{10} z_2^{17} z_3^{10} + a_{59} z_1^{11} z_2^{17} z_3^9 + a_{171} z_1^{20} z_3^{17} = 0.$$

Now apply the transformation V five successive times. The result is

$$(62) \quad z_2^{85} (a_{148} z_1 + a_{171} z_3) + a_{52} z_1^8 z_2^{51} z_3^{27} + a_{26} z_1^{11} z_2^{34} z_3^{41} \\ + a_{99} z_1^4 z_2^{68} z_3^{14} + a_9 z_1^{14} z_2^{17} z_3^{55} + a_{59} z_1^7 z_2^{51} z_3^{28} \\ + a_1 z_1^{17} z_2^{69} = 0.$$

One application of V and then six successive applications of U gives

$$\begin{aligned}
 (63) \quad & z_2^{51} (a_{52} z_1 + a_{148} z_3) + a_{26} z_1^{16} z_2^{34} z_3^2 + a_{99} z_1^{15} z_2^{34} z_3^3 \\
 & + a_9 z_1^{31} z_2^{17} z_3^4 + a_{59} z_1^{30} z_2^{17} z_3^5 + a_1 z_1^{46} z_3^6 \\
 & + a_{171} z_1^{29} z_2^{17} z_3^6 = 0.
 \end{aligned}$$

All this indicates that the curves of (60) have in common in the neighborhood of O_2 :

- (a) two successive triple points O_{21} and O_{211} ;
- (b) a double point O_{23} ;
- (c) ten simple points $O_{233}, O_{2333}, O_{23333}, O_{233333}, O_{231},$
 $O_{2311}, O_{23111}, O_{231111}, O_{2311111},$ and $O_{23111111}.$

Therefore, two curves of the system (60) intersect $1 \cdot 11^2 + 2 \cdot 3^2 + 1 \cdot 2^2 + 10 \cdot 1^2$ or 153 times at O_2 . Thus, the system (60) has degree $289 - 153$ or 136. The sum of the multiplicities of the points O'_{10} for ϕ , O'_8 for ϕ_1 , and O'_6 for ϕ_2 must be $153/17$ or 9. Hence O'_6 is multiple of order one for ϕ_2 .

Now apply the quadratic transformation U twice in succession to the projectivity obtained from (60). Then substitute $z_1 = k z_3$ and observe that the limit as z_3 goes to zero is

$$(64) \quad \left\{ \begin{array}{l} \left\| \begin{array}{ccc} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{array} \right\| = 0, \\ X_5 = X_7 = X_8 = X_9 = X_{10} = X_{11} = 0. \end{array} \right.$$

Hence to the points infinitely near O_{211} correspond certain points on ϕ_2 of (64) near O_6' . Note that (64) is the projection of (56) to the space $X_8 = 0$.

Now apply V five successive times to the projectivity and substitute $z_3 = k z_1$. As z_1 tends to the limit zero the result is

$$(65) \quad \begin{cases} X_{11} = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = X_8 = X_{10} = 0. \end{cases}$$

Note that (65) is the projection of (57) to the space $X_8 = 0$.

Apply V to the projectivity, and then apply U five successive times. Substitute $z_3 = k z_1$ and take the limit as z_1 tends to zero; the result is

$$(66) \quad \begin{cases} X_9 = k X_4 \\ X_1 = X_2 = X_3 = X_5 = X_7 = X_8 = X_{10} = X_{11} = 0. \end{cases}$$

Surfaces (64) and (65) are projections of previous tangent elements. Thus, the additional tangent element to ϕ_2 is the plane (66) as stated below.

Theorem 5: The surface ϕ_2 has a new tangent element

$$(67) \quad \begin{cases} X_9 = k X_4 \\ X_1 = X_2 = X_3 = X_5 = X_7 = X_8 = X_{10} = X_{11} = 0 \end{cases}$$

at the point O_6' .

3. Surface ϕ_3

Project the surface ϕ_2 from the point $O'_6 (0,0,0,0,0,1,0,0,0,0,0)$ onto the space $X_6 = 0$ to obtain the surface, getting

$$(\phi_3) \quad \left\{ \begin{array}{l} \left\| \begin{array}{cccccc} X_4 & X_9 X_5 & X_3 & X_7 & X_2 X_5 & X_2^2 \\ X_7 & X_7 X_{11} & X_5 & X_{11} & X_1 X_{11} & X_1 X_5 \end{array} \right\| = 0, \\ X_{10} = X_8 = X_6 = 0. \end{array} \right.$$

Two members of the family (60) intersect in $1 \cdot 11^2 + 2 \cdot 3^2 + 1 \cdot 2^2 + 10 \cdot 1^2$ or 153 fixed points. Thus, the order of ϕ_3 is $(289 - 153)/17$ or 8.

Consider the family of curves which pass through $O_2 (0, 1, 0)$,

$$(68) \quad a_{148} x_1^5 x_2^5 x_3^{11} + a_{26} x_1^{11} x_2^4 x_3^2 + a_{99} x_1^4 x_2^3 x_3^{10} + a_9 x_1^{14} x_2^2 x_3 \\ + a_{59} x_1^7 x_2 x_3^9 + a_1 x_1^{17} + a_{171} x_1^{17} = 0.$$

Repeated application of the transformations U and V will show that members of the family (68) have in common at O_2 :

- (a) one triple point O_{21} ;
- (b) one double point O_{211} ;
- (c) thirteen simple points $O_{213}, O_{2133}, \dots, O_{213(8)}, O_{23},$
 $O_{233}, O_{2333}, O_{23333},$ and $O_{233333}.$

Hence, two members of the system (68) intersect in $1 \cdot 12^2 + 1 \cdot 3^2 + 1 \cdot 2^2 + 13 \cdot 1^2$ or 170 fixed points at O_2 . Therefore, the system (68) has degree $289 - 170$ or 119. This indicates that the sum of the multiplicities of O'_{10} for ϕ , O'_8 for ϕ_1 , O'_6 for ϕ_2 , and

O_4^1 for ϕ_3 is 170/17 or 10. Thus, O_4^1 is of multiplicity one for ϕ_3 .

In a manner very much like those used before, apply the quadratic transformations U and V repeatedly to the projectivity and observe that:

(a) certain points near O_4^1 on ϕ_3 situated on

$$(69) \quad \begin{cases} X_2^2 - X_1 X_3 = 0 \\ X_m = 0, \quad (m = 5, 6, 7, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211} ;

(b) certain other neighborhood points near O_4^1 on ϕ_3 situated on

$$(70) \quad \begin{cases} X_{11} = k X_9 \\ X_m = 0, \quad (m = 1, 2, 3, 5, 6, 7, 8, 10) \end{cases}$$

correspond to the points infinitely near O_{233333} ;

(c) and similarly, other points on ϕ_3 situated on

$$(71) \quad \begin{cases} X_3 = k X_9 \\ X_m = 0, \quad (m = 1, 2, 5, 6, 7, 8, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{213(8)}$.

Note that (69) is the projection of (64) to the space $X_6 = 0$ and that (70) is the projection of (65) to the space $X_6 = 0$.

The new tangent element is the plane projecting (71) from the point O_4' . Hence, the following

Theorem 6: The surface ϕ_3 has a new tangent element

$$(72) \quad \begin{cases} X_3 = k X_9 \\ X_m = 0, \quad (m = 1, 2, 5, 6, 7, 8, 10, 11) \end{cases}$$

at the point O_4' .

4. Surface ϕ_4

Project the surface ϕ_3 from the point O_4' (0,0,0,1,0,0,0,0,0,0,0) to the space $X_4 = 0$ to obtain the surface

$$(\phi_4) \quad \left\{ \begin{array}{l} \left\| \begin{array}{ccccc} X_9 X_5 & X_3 & X_7 & X_2 X_5 & X_2^2 \\ X_7 X_{11} & X_5 & X_{11} & X_1 X_{11} & X_1 X_5 \end{array} \right\| = 0, \\ X_{10} = X_8 = X_6 = X_4 = 0. \end{array} \right.$$

Two members of the family (68) intersect in $1 \cdot 12^2 + 1 \cdot 3^2 + 1 \cdot 2^2 + 13 \cdot 1^2$ or 170 fixed points at O_2 (0, 1, 0). Thus the order of ϕ_4 is $(289 - 170)/17$ or 7.

Examine the family of curves which pass through O_2 ,

$$(73) \quad a_{26} x_1^{11} x_2^4 x_3^2 + a_{99} x_1^4 x_2^3 x_3^{10} + a_9 x_1^{14} x_2^2 x_3 + a_{59} x_1^7 x_2 x_3^9 \\ + a_1 x_1^{17} + a_{171} x_3^{17} = 0.$$

Successive applications of the transformation U and V indicate that the members of the family (73) have in common at O_2 :

- (a) one four tuple point O_{23} ;
- (b) three double points O_{21} , O_{211} , and O_{231} ;
- (c) seven simple points O_{2313} , O_{23133} , O_{2311} , O_{23111} , O_{231111} , $O_{2311111}$, and $O_{231(6)}$.

Thus, two members of the system (73) meet at O_2 in $1 \cdot 13^2 + 1 \cdot 4^2 + 3 \cdot 2^2 + 7 \cdot 1^2$ or 204 fixed points, and the system has degree $289 - 204$ or 85. This indicates that the sum of the orders of O'_{10} for ϕ , O'_8 for ϕ_1 , O'_6 for ϕ_2 , O'_4 for ϕ_3 , and O'_9 for ϕ_4 is $204/17$ or 12. Consequently, O'_9 for ϕ_4 is multiple of order two.

As before, apply the quadratic transformations to the projectivity, then make the necessary substitution of $z_1 = k z_3$ or $z_3 = k z_1$, and take the limit as z_3 or z_1 tends to zero respectively. This gives:

- (a) certain points near O'_9 on ϕ_4 situated on

$$(74) \quad \begin{cases} X_2^2 - X_1 X_3 = 0 \\ X_m = 0, \quad (m = 4, 5, 6, 7, 8, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211} ;

- (b) similarly, certain points on ϕ_4 situated on

$$(75) \quad \begin{cases} X_3 = k X_7 \\ X_m = 0, \quad (m = 1, 2, 4, 5, 6, 8, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{23111111}$,

(c) and other points on ϕ_4 situated on

$$(76) \quad \begin{cases} X_{11} = k X_7 \\ X_m = 0, \quad (m = 1, 2, 3, 4, 5, 6, 8, 10) \end{cases}$$

correspond to the points infinitely near O_{23133} .

The curve (74) is the projection of (69) to the space $X_4 = 0$.

Hence, the following

Theorem 7: The surface ϕ_4 has the two new tangent elements

$$(77) \quad \begin{cases} X_3 = k X_7 \\ X_m = 0, \quad (m = 1, 2, 4, 5, 6, 8, 10, 11) \end{cases}$$

and

$$(78) \quad \begin{cases} X_{11} = k X_7 \\ X_m = 0, \quad (m = 1, 2, 3, 4, 5, 6, 8, 10) \end{cases}$$

at the point O'_9 .

5. Surface ϕ_5

Project the surface ϕ_4 from the point $O^i_9 (0,0,0,0,0,0,0,0,1,0,0)$ to the space $X_9 = 0$ to obtain the surface

$$(\phi_5) \quad \begin{cases} \left| \begin{vmatrix} X_3 & X_7 & X_2 X_5 & X_2^2 \\ X_5 & X_{11} & X_1 X_{11} & X_1 X_5 \end{vmatrix} \right| = 0, \\ X_{10} = X_8 = X_6 = X_4 = X_9 = 0. \end{cases}$$

Two members of family (73) intersected in 204 fixed points at O_2 . Thus the order of ϕ_5 is $(289 - 204)/17$ or 5.

Applications of the transformations U and V indicate that the family

$$(79) \quad a_{99} x_1^4 x_2^3 x_3^{10} + a_9 x_1^{14} x_2^2 x_3 + a_{59} x_1^7 x_2 x_3^9 + a_1 x_1^{17} + a_{171} x_3^{17} = 0,$$

has a group of multiple points at O_2 . These multiple points are:

(a) one triple point O_{23} ;

(b) one double point O_{21} ;

(c) twelve simple points O_{231} , O_{2313} , O_{23133} , O_{211} ,

O_{213} , O_{2133} , ..., $O_{213(7)}$ and $O_{213(8)}$.

Hence, two members of the system (79) have in common at O_2 , $1 \cdot 14^2 + 1 \cdot 3^2 + 1 \cdot 2^2 + 12 \cdot 1^2$ or 221 fixed points, and the degree of the system is $289 - 221$ or 68. Delete from $221/17$ the sum of the orders of O'_{10} , O'_8 , O'_6 , O'_4 , and O'_9 . Thus, the point O'_3 is of multiplicity one for ϕ_5 .

Applications of U and V to the projectivity give:

(a) the points on ϕ_5 situated on

$$(80) \quad \begin{cases} X_1 = k X_2 \\ X_m = 0, \quad (m = 4, 5, 6, 7, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211} ;

(b) the points on ϕ_5 situated on

$$(81) \quad \begin{cases} X_{11} = k X_7 \\ X_m = 0, \quad (m = 1, 2, 4, 5, 6, 8, 9, 10) \end{cases}$$

correspond to the points infinitely near O_{23133} ;

(c) the points on ϕ_5 situated on

$$(82) \quad \begin{cases} X_2 = k X_7 \\ X_m = 0, \quad (m = 1, 4, 5, 6, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{213(8)}$.

Note that the surface (80) is the projection of (74) to the space $X_9 = 0$, and that (81) is the projection of (76) to the same space.

Hence, the following

Theorem 8: The surface ϕ_5 has a tangent new element

$$(83) \quad \begin{cases} X_2 = k X_7 \\ X_m = 0, \quad (m = 1, 4, 5, 6, 8, 9, 10, 11) \end{cases}$$

at the point O_3^1 .

6. Surface ϕ_6

Project the surface ϕ_5 from the point $O_3^1 (0,0,1,0,0,0,0,0,0,0)$

to the space $X_3 = 0$ to obtain the surface

$$(\phi_6) \quad \begin{cases} \left\| \begin{array}{ccc} X_7 & X_2 X_5 & X_2^2 \\ X_{11} & X_1 X_{11} & X_1 X_5 \end{array} \right\| = 0, \\ X_{10} = X_8 = X_6 = X_4 = X_9 = X_3 = 0. \end{cases}$$

Two members of the family (79) intersected in 221 variable points at O_2 . Thus, the order of ϕ_6 is $(289 - 221)/17$ or 4.

Now use the transformations U and V to find the multiple points at O_2 of the family

$$(84) \quad a_9 x_1^{14} x_2^2 x_3 + a_{59} x_1^7 x_2 x_3^9 + a_1 x_1^{17} + a_{171} x_3^{17} = 0.$$

The multiple points are:

(a) seven double points O_{23} , O_{231} , \dots , $O_{23(5)}$, and $O_{23(6)}$;

(b) two simple points O_{21} and O_{211} .

Consequently, two members of the system (84) have in common at O_2 , $1 \cdot 15^2 + 7 \cdot 2^2 + 2 \cdot 1^2$ or 255 fixed points, and the degree of the system is $289 - 255$ or 34. Since the sum of the orders of O_{10}' , O_8' , O_6' , O_4' , O_9' , and O_3' is 13, the value $255/17$, or 15, implies that the multiplicity of O_7' for the surface ϕ_6 is 2.

Applications of U and V to the projectivity give:

(a) the points on ϕ_6 situated on

$$(85) \quad \begin{cases} X_1 = k X_2 \\ X_m = 0, \quad (m = 3, 4, 5, 6, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211} ;

(b) the points on ϕ_6 situated on

$$(86) \quad \begin{cases} X_5^2 - X_2 X_{11} = 0 \\ X_m = 0, \quad (m = 1, 3, 4, 6, 8, 9, 10) \end{cases}$$

correspond to the points infinitely near $O_{23(6)}$.

The surface (85) is the projection of the surface (80) to the space $X_3 = 0$. Hence, the following

Theorem 9: The surface ϕ_6 has a new tangent element

$$(87) \quad \begin{cases} X_5^2 - X_2 X_{11} = 0 \\ X_m = 0, \quad (m = 1, 3, 4, 6, 8, 9, 10) \end{cases}$$

at the point O_7^1 .

7. Surface ϕ_7

The surface ϕ_6 projects from the point O_7^1 (0,0,0,0,0,0,1,0,0,0,0) to the space $X_7 = 0$ to a new surface

$$(\phi_7) \quad \begin{cases} X_5^2 - X_2 X_{11} = 0 \\ X_{10} = X_8 = X_6 = X_4 = X_9 = X_3 = X_7 = 0. \end{cases}$$

Two members of the family (84) intersected in 255 fixed points at O_2 .

Thus, the order of ϕ_7 is $(289 - 255)/17$ or 2.

The transformations U and V establish the multiple points at O_2 for the family

$$(88) \quad a_{59} x_1^7 x_2 x_3^9 + a_1 x_1^{17} + a_{171} x_3^{17} = 0.$$

These are sixteen simple points $O_{21}, O_{213}, O_{2133}, \dots, O_{213(8)}, O_{23}, O_{231}, O_{2311}, \dots, O_{231(5)},$ and $O_{231(6)}$. Thus, two members of the system (88) have in common at $O_2, 1 \cdot 16^2 + 16 \cdot 1^2$ or 272 fixed points, and the degree of the system is $289 - 272$ or 17. The multiplicities of O_{10}^1 ,

$O'_8, O'_6, O'_4, O'_9, O'_3, O'_7$, and O'_2 total to $272/17$ or 16. Therefore O'_2 is a simple point for ϕ_7 .

Apply U and V to the projectivity and observe that:

(a) the points on ϕ_7 situated on

$$(89) \quad \begin{cases} X_5 = k X_{11} \\ X_m = 0, \quad (m = 1, 3, 4, 6, 7, 8, 9, 10) \end{cases}$$

correspond to the points infinitely near $O_{23111111}$;

(b) the points on ϕ_7 situated on

$$(90) \quad \begin{cases} X_1 = k X_5 \\ X_m = 0, \quad (m = 3, 4, 6, 7, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{213(8)}$.

The surface (89) is the projection of (86) to the space

$X_7 = 0$. Hence, the following

Theorem 10: The surface ϕ_7 has a new tangent element

$$\begin{cases} X_1 = k X_5 \\ X_m = 0, \quad (m = 3, 4, 6, 7, 8, 9, 10, 11) \end{cases}$$

at the point O'_2 .

8. Summary

A sequence of projected surfaces was described and the tangent elements investigated. The orders of these surfaces were arrived at and the multiplicities of the points calculated.

The following figures are an outline of the multiple points that the generating curves have at the O_2 vertex of the triangle of reference in the plane.

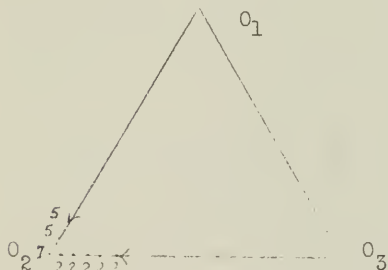


Figure 2

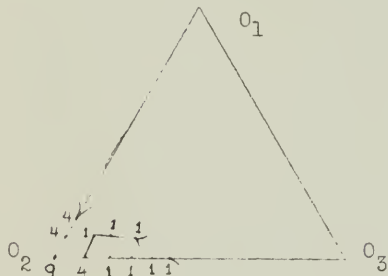


Figure 3

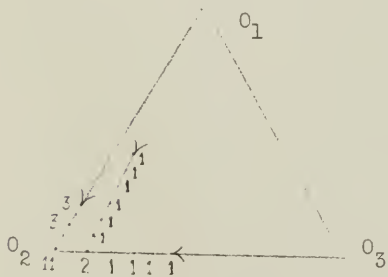


Figure 4

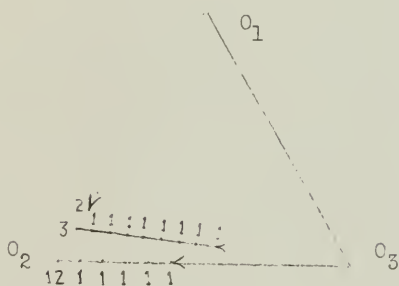


Figure 5

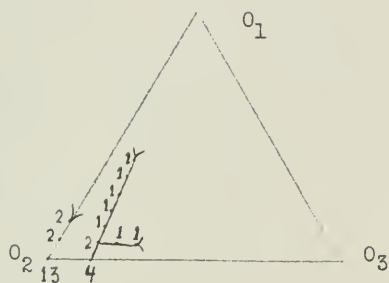


Figure 6

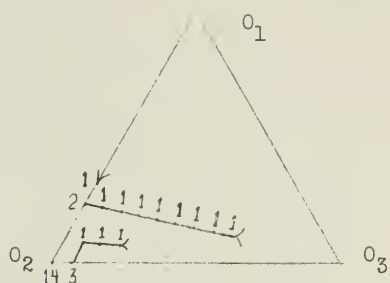


Figure 7

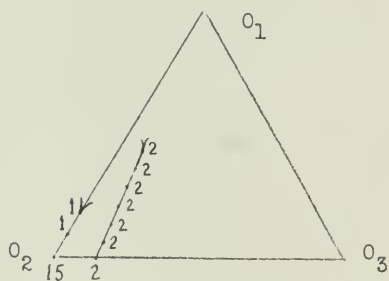


Figure 8

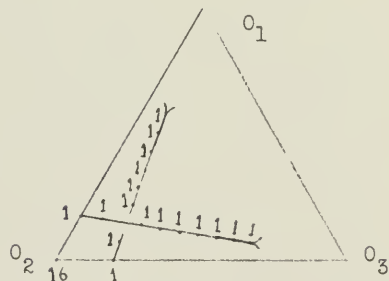


Figure 9

The following chart lists the various results of this chapter and some information on ϕ from Chapter II.

Surface	Order of Surface	Point on Surface	Multiplicity of Point	Tangent Element to Surface at Point
ϕ	17	O'_{10}	7	$\begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_6 \\ X_2 & X_3 & X_4 & X_6 & X_8 \end{vmatrix} = 0$ <p>and</p> $X_m = 0, (m = 5, 7, 9, 11)$ $X_9^2 - X_8 X_{11} = 0$ <p>and</p> $X_m = 0, (m = 1, 2, 3, 4, 5, 6, 7)$
ϕ_1	10	O'_8	1	$X_6 = k X_9$ <p>and</p> $X_m = 0, (m = 1, 2, 3, 4, 5, 7, 10, 11)$
ϕ_2	9	O'_6	1	$X_9 = k X_4$ <p>and</p> $X_m = 0, (m = 1, 2, 3, 5, 7, 8, 10, 11)$
ϕ_3	8	O'_4	1	$X_3 = k X_9$ <p>and</p> $X_m = 0, (m = 1, 2, 5, 6, 7, 8, 10, 11)$

Surface	Order of Surface	Point on Surface	Multiplicity of Point	Tangent Element to Surface at Point
ϕ_4	7	O'_9	2	$X_3 = k X_9$ and $X_m = 0, (m = 1, 2, 4, 5, 6, 8, 10, 11)$ $X_{11} = k X_7$ and $X_m = 0, (m = 1, 2, 3, 4, 5, 6, 8, 10)$
ϕ_5	5	O'_3	1	$X_2 = k X_7$ and $X_m = 0, (m = 1, 4, 5, 6, 8, 9, 10, 11)$
ϕ_6	4	O'_7	2	$X_5^2 - X_2 X_{11} = 0$ and $X_m = 0, (m = 1, 3, 4, 6, 8, 9, 10)$
ϕ_7	2	O'_2	1	$X_1 = k X_5$ and $X_m = 0, (m = 3, 4, 6, 7, 8, 9, 10, 11)$

The author realizes that the results obtained in this chapter only begin to identify the information that is obtainable about the projected surfaces. Further study will undoubtedly yield many other fascinating truths, illuminating the facts of this chapter.

CHAPTER IV

A RATIONAL SURFACE F IN S_{11}

To a certain plane curve shown below, of order seventeen, and which is not invariant under H corresponds on \mathbb{P} a curve of order two hundred eighty-nine. This curve is cut out on \mathbb{P} by a seventeenth order hypersurface. Furthermore, the coefficients of the equations of the latter surface are functions of the coefficients of the equation of the plane curve considered.

In order to see this, consider the plane curve of order seventeen,

$$(92) \quad \theta_1 \equiv \sum c_{i,j,k} x_1^i x_2^j x_3^k = 0,$$

where

$$i + j + k = 17.$$

Apply H sixteen times in succession to (92); this gives

$$\theta_n = \sum E^{w(n)} c_{i,j,k} x_1^i x_2^j x_3^k = 0,$$

where

$$i + j + k = 17$$

$$n = 2, 3, \dots, 17,$$

and $w(n)$ is the remainder when $(n - 1)(j + 15k)$ is divided by 17.

The curve,

$$(93) \quad \theta_1 \theta_2 \theta_3 \dots \theta_{17} = 0,$$

corresponds to a curve C on \mathbb{P} , where C is in birational correspondence with each of the curves $\theta_m = 0$ ($m = 1, 2, \dots, 17$). That is, to a point

of C corresponds seventeen points of the plane with one of the seventeen points on each of the seventeen curves considered.

The curve (93) meets a curve of (1) in two hundred eighty-nine groups of I_{17} . This implies that the hyperplane related to (1) intersects C in 289 points. Hence, C is of order 289.

Let us vary θ_1 in a continuous manner in its plane until its equation becomes equal to (1). The corresponding C varies on ϕ and reduces to the section of ϕ by the hyperplane,

$$(94) \quad a_1 X_1 + a_9 X_2 + a_{26} X_3 + a_{52} X_4 + a_{59} X_5 + a_{87} X_6 + a_{99} X_7 \\ + a_{133} X_8 + a_{148} X_9 + a_{170} X_{10} + a_{171} X_{11} = 0,$$

counted seventeen times. That is, the section of ϕ is made by the reducible hypersurface of order 17,

$$(95) \quad (a_1 X_1 + a_9 X_2 + a_{26} X_3 + \dots + a_{171} X_{11})^{17} = 0.$$

This implies that the curves C are cut out on ϕ by seventeenth order hypersurfaces.

Now, consider $\theta_1 = 0$ varying in the plane and becoming equation (2). The curve (93) becomes

$$(96) \quad (g(x_1, x_2, x_3))^{17} = 0,$$

and the curve C becomes a curve A counted seventeen times. Consequently, A must be cut out on ϕ by a hypersurface of order seventeen.

By simplifying (96) and applying T one arrives at the following equation for the hypersurface

$$(97) \quad \Psi(X_1, X_2, \dots, X_{11}) = (g(x_1, x_2, x_3))^{17} = a_{17}^{17} X_1^{12} X_{11}^5 \\ + a_{17}^{16} a_{29} X_1^{11} X_2 X_9 X_{11}^4 + \dots + a_{169}^{17} X_{10}^8 X_{11}^9 = 0.$$

The fact that the x_i 's ($i = 1, 2, 3$) group together into factors of X_i ($i = 1, \dots, 11$) can be demonstrated by solving certain equations relating the exponents of x_i ($i = 1, 2, 3$) obtained from possible powers of terms of $g(x_1, x_2, x_3)$ to the exponents of x_i ($i = 1, 2, 3$) obtained from possible factors of X_i ($i = 1, \dots, 11$).

Take a surface F in S_{11} whose equations are:

$$(F) \quad \begin{aligned} X_{12}^{17} &= \Psi(X_1, X_2, \dots, X_{11}) \\ \left\| \begin{array}{cccccccc} X_{10}X_1 & X_8X_5 & X_6 & X_4 & X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_4X_6 & X_7X_9 & X_9 & X_7 & X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{array} \right\| &= 0. \end{aligned}$$

Now the author demonstrates that F is a rational surface. To do this, a projective correspondence is set up between the plane and F using the following transformation T' .

$$\begin{aligned}
(T') \quad \frac{x_1}{x_1^{17}} &= \frac{x_2}{x_1^{14} x_2^2 x_3} = \frac{x_3}{x_1^{11} x_2^4 x_3^2} = \frac{x_4}{x_1^8 x_2^6 x_3^3} = \frac{x_5}{x_1^7 x_2 x_3^9} \\
&= \frac{x_6}{x_1^5 x_2^8 x_3^4} = \frac{x_7}{x_1^4 x_2^3 x_3^{10}} = \frac{x_8}{x_1^2 x_2^{10} x_3^5} = \frac{x_9}{x_1 x_2^5 x_3^{11}} = \frac{x_{10}}{x_2^{17}} \\
&= \frac{x_{11}}{x_3^{17}} = \frac{x_{12}}{g(x_1, x_2, x_3)} = \frac{1}{\rho}.
\end{aligned}$$

This transformation orders to each point of the plane a unique point of F . It needs to be shown that the converse is true. A development of T'^{-1} will show this and thus show that F is a rational surface.

The first of the following ten equations comes directly from T' . The others are derived with successive multiplications by $x_4 x_5 x_6$ and applications of T' .

$$(98) \quad g(x_1, x_2, x_3) = \rho x_{12}$$

$$\begin{aligned}
(99) \quad & a_{124} x_4 x_5 x_6 x_1^2 x_2 x_3^{14} + a_{169} x_4 x_5 x_6 x_2^8 x_3^9 \\
& + a_{42} x_3 x_4 x_5 x_1^3 x_2^6 x_3^8 + a_{75} x_4 x_6 x_9 x_1^{12} x_3^5 \\
& + a_{96} x_4 x_6 x_9 x_1^{10} x_2^7 + a_{143} x_4 x_6 x_8 x_1^6 x_2^4 x_3^7 \\
& + a_{119} x_3 x_5 x_9 x_1^4 x_2^{11} x_3^2 + a_{17} x_2 x_4 x_9 x_1^9 x_2^2 x_3^6 \\
& + a_{29} x_2 x_4 x_9 x_1^7 x_2^9 x_3 + a_{58} x_3 x_4 x_5 x_1 x_2^{13} x_3^3 \\
& = \rho x_4 x_5 x_6 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(100) \quad & a_{124} x_4^2 x_5 x_6^2 x_{11} x_1^9 x_2^2 x_3^6 + a_{169} x_4^2 x_5 x_6^2 x_{11} x_1^7 x_2^9 x_3 \\
& + a_{75} x_1 x_4^2 x_6^2 x_9 x_1^2 x_2 x_3^{14} + a_{96} x_1 x_4^2 x_6^2 x_9 x_2^8 x_3^9 \\
& + a_{143} x_4^2 x_6^2 x_8 x_9 x_1^{12} x_3^5 + a_{17} x_2^2 x_4^2 x_5 x_{11} x_1 x_2^{13} x_3^3 \\
& + a_{29} x_3^2 x_4^2 x_9^2 x_1^{10} x_2^7 + a_{58} x_3 x_4^2 x_5^2 x_{10} x_1^6 x_2^4 x_3^7 \\
& + a_{42} x_2 x_4^2 x_5^2 x_9 x_1^4 x_2^{11} x_3^2 + a_{119} x_4^2 x_5^2 x_6^2 x_1^3 x_2^6 x_3^8 \\
& = \rho x_4^2 x_5^2 x_6^2 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(101) \quad & a_{124} x_2 x_4^3 x_5 x_6^3 x_{11} x_1^2 x_2 x_3^{14} + a_{169} x_2 x_4^3 x_5 x_6^3 x_{11} x_2^8 x_3^9 \\
& + a_{75} x_1 x_4^3 x_6^3 x_9 x_{11} x_1^9 x_2^2 x_3^6 + a_{96} x_1 x_4^3 x_5 x_6^3 x_{11} x_1 x_2^{13} x_3^3 \\
& + a_{143} x_4^3 x_5^2 x_6^3 x_9 x_1^7 x_2^9 x_3 + a_{17} x_2^2 x_4^3 x_5^2 x_{10} x_{11} x_1^6 x_2^4 x_3^7 \\
& + a_{29} x_3^3 x_4^3 x_5 x_9^2 x_1^4 x_2^{11} x_3^2 + a_{58} x_3 x_4^3 x_5^3 x_9 x_{10} x_1^{10} x_2^7 \\
& + a_{42} x_2 x_4^3 x_5^3 x_6 x_8 x_1^3 x_2^6 x_3^8 + a_{119} x_4^2 x_5^3 x_6^2 x_8 x_1^{12} x_3^5 \\
& = \rho x_4^3 x_5^3 x_6^3 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(102) \quad & a_{124} x_2 x_4^4 x_5 x_6^4 x_{11}^2 x_1^9 x_2^2 x_3^6 + a_{169} x_2 x_4^4 x_5 x_6^4 x_{11}^2 x_1^7 x_2^9 x_3 \\
& + a_{75} x_1 x_2 x_4^4 x_6^4 x_9 x_{11} x_1^2 x_2 x_3^{14} \\
& + a_{96} x_1 x_4^4 x_5 x_6^4 x_8 x_{11} x_1^6 x_2^4 x_3^7 \\
& + a_{143} x_4^4 x_5^2 x_6^4 x_8 x_9 x_1^{12} x_3^5 + a_{17} x_2^2 x_4^4 x_5^4 x_{10} x_{11} x_1^4 x_2^{11} x_3^2 \\
& + a_{29} x_3^3 x_4^4 x_5^3 x_9 x_{10} x_1^3 x_2^6 x_3^8 + a_{58} x_2 x_3 x_4^4 x_5^4 x_9 x_{10} x_1 x_2^{13} x_3^3 \\
& + a_{42} x_2 x_3 x_4^4 x_5^4 x_8^2 x_2^8 x_3^9 + a_{119} x_3 x_4^3 x_5^4 x_6^2 x_8^2 x_9 x_1^{10} x_2^7 \\
& = \rho x_4^4 x_5^4 x_6^4 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(103) \quad & a_{124} x_2^2 x_4^5 x_5 x_6^5 x_{11}^2 x_1^2 x_2 x_3^{14} + a_{169} x_2^2 x_4^5 x_5 x_6^5 x_{11}^2 x_2^8 x_3^9 \\
& + a_{75} x_1 x_2 x_4^5 x_6^5 x_9 x_{11}^2 x_1^9 x_2^2 x_3^6 \\
& + a_{96} x_1 x_4^5 x_5 x_6^5 x_8 x_9 x_{11} x_1^{12} x_3^5 \\
& + a_{143} x_1 x_4^5 x_5^2 x_6^5 x_9 x_{11} x_1^4 x_2^{11} x_3^2 \\
& + a_{17} x_2^2 x_4^5 x_5^5 x_8 x_{10} x_{11} x_1^7 x_2^9 x_3 \\
& + a_{29} x_3^3 x_4^5 x_5^5 x_9 x_{10} x_1 x_2^{13} x_3^3 + a_{58} x_2 x_3 x_4^5 x_5^5 x_9 x_{10}^2 x_1^6 x_2^4 x_3^7 \\
& + a_{42} x_2 x_3 x_4^5 x_5^5 x_8^3 x_1^3 x_2^6 x_3^8 + a_{119} x_3 x_4^4 x_5^5 x_6^2 x_8^2 x_9 x_1^{10} x_2^7 \\
& = \rho x_4^5 x_5^5 x_6^5 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(104) \quad & a_{124} x_2^2 x_4^6 x_5 x_6^6 x_{11}^3 x_1^9 x_2^2 x_3^6 + a_{169} x_2^2 x_4^6 x_5 x_6^6 x_{11}^3 x_1^7 x_2^9 x_3 \\
& + a_{75} x_1 x_2^2 x_4^6 x_6^6 x_9 x_{11}^2 x_1^2 x_2 x_3^{14} \\
& + a_{96} x_1^2 x_4^6 x_5 x_6^6 x_9 x_{11}^2 x_1^4 x_2^{11} x_3^2 \\
& + a_{143} x_1 x_4^6 x_5^2 x_6^6 x_9^2 x_{11} x_1^{10} x_2^7 + a_{17} x_2^2 x_4^6 x_5^6 x_8 x_{10}^2 x_{11} x_1^{12} x_3^5 \\
& + a_{29} x_3^3 x_4^6 x_5^6 x_9 x_{10}^2 x_1^6 x_2^4 x_3^7 + a_{58} x_2 x_3^2 x_4^6 x_5^6 x_9 x_{10}^2 x_2^8 x_3^9 \\
& + a_{42} x_2^2 x_4^6 x_5^6 x_8^4 x_1^3 x_2^6 x_3^8 + a_{119} x_2 x_3 x_4^5 x_5^6 x_6^2 x_8^2 x_9 x_1 x_2^{13} x_3^3 \\
& = \rho x_4^6 x_5^6 x_6^6 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(105) \quad & a_{124} x_2^3 x_4^7 x_5 x_6^7 x_{11}^3 x_1^2 x_2 x_3^{14} + a_{169} x_2^3 x_4^7 x_5 x_6^7 x_{11}^3 x_2^8 x_3^9 \\
& + a_{75} x_1 x_2^2 x_4^7 x_6^7 x_9 x_{11}^3 x_1^9 x_2^2 x_3^6 + a_{96} x_1^2 x_4^7 x_5 x_6^7 x_9^2 x_{11}^2 x_1^{10} x_2^7 \\
& + a_{143} x_1 x_3 x_4^7 x_5^2 x_6^7 x_9^2 x_{11} x_1^6 x_2^4 x_3^7 \\
& + a_{17} x_2^3 x_4^7 x_5^7 x_8 x_{10}^2 x_{11} x_1^3 x_2^6 x_3^8 + a_{29} x_3^3 x_4^7 x_5^7 x_{10}^3 x_{11} x_1^{12} x_3^5 \\
& + a_{58} x_2 x_3^2 x_4^7 x_5^7 x_9^2 x_{10}^2 x_1^4 x_2^{11} x_3^2 + a_{42} x_2^2 x_4^7 x_5^7 x_8^4 x_9 x_1^7 x_2^9 x_3 \\
& + a_{119} x_1 x_3 x_4^6 x_5^7 x_6^2 x_8^3 x_9 x_1 x_2 x_3^{13} \\
& = \rho x_4^7 x_5^7 x_6^7 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(106) \quad & a_{124} x_2^3 x_4^8 x_5 x_6^8 x_{11}^4 x_1^9 x_2^2 x_3^6 + a_{169} x_2^3 x_4^8 x_5 x_6^8 x_{11}^4 x_1^7 x_2^9 x_3 \\
& + a_{75} x_1 x_2^3 x_4^8 x_6^8 x_9 x_{11}^3 x_1^2 x_2 x_3^{14} \\
& + a_{96} x_1^3 x_4^8 x_5 x_6^8 x_9^2 x_{11}^2 x_2^8 x_3^9 \\
& + a_{143} x_1 x_3 x_4^8 x_5^2 x_6^8 x_9 x_{11} x_1^{12} x_3^5 \\
& + a_{17} x_2^3 x_4^8 x_5^8 x_8^2 x_{10}^2 x_{11} x_1^6 x_2^4 x_3^7 \\
& + a_{29} x_2 x_3^3 x_4^8 x_5^8 x_{10}^3 x_{11} x_1^3 x_2^6 x_3^8 \\
& + a_{58} x_3^3 x_4^8 x_5^8 x_8 x_9^2 x_{10}^2 x_1^{10} x_2^7 + a_{42} x_2^2 x_3 x_4^8 x_5^8 x_8^4 x_9 x_1 x_2^{13} x_3^3 \\
& + a_{119} x_1 x_3 x_4^7 x_5^8 x_6^2 x_8^4 x_9 x_1^4 x_2^{11} x_3^2 = \rho x_4^8 x_5^8 x_6^8 x_{12}.
\end{aligned}$$

$$\begin{aligned}
(107) \quad & a_{124} x_2^4 x_4^9 x_5 x_6^9 x_{11}^4 x_1^2 x_2 x_3^{14} + a_{169} x_2^4 x_4^9 x_5 x_6^9 x_{11}^4 x_2^8 x_3^9 \\
& + a_{75} x_1 x_2^3 x_4^9 x_6^9 x_9 x_{11}^4 x_1^9 x_2^2 x_3^6 \\
& + a_{96} x_1^3 x_4^9 x_5 x_6^9 x_9^2 x_{11}^3 x_1^7 x_2^9 x_3 + a_{143} x_1^2 x_4^9 x_5^2 x_6^9 x_9^4 x_{11} x_1^{12} x_3^5 \\
& + a_{17} x_2^3 x_4^9 x_5^9 x_8^2 x_9 x_{10}^2 x_{11} x_1^{10} x_2^7 \\
& + a_{29} x_2 x_3^3 x_4^9 x_5^9 x_8 x_{10}^3 x_{11} x_1^6 x_2^4 x_3^7 \\
& + a_{58} x_3^4 x_4^9 x_5^9 x_8 x_9^2 x_{10}^2 x_1^4 x_2^{11} x_3^2 \\
& + a_{42} x_2^3 x_4^9 x_5^9 x_8^4 x_9 x_{10} x_1^3 x_2^6 x_3^8 \\
& + a_{119} x_1 x_3 x_4^9 x_5^9 x_6^2 x_8^4 x_9 x_1 x_2^{13} x_3^3 = \rho x_4^9 x_5^9 x_6^9 x_{12}.
\end{aligned}$$

Note that the previous ten equations are linear in $x_1^{12} x_3^5$, $x_1^{10} x_2^7$, ... , and $x_2^8 x_3^9$, i.e., the terms of $g(x_1, x_2, x_3)$. Thus, Cramer's rule can be easily used to solve for these expressions.

The ten equations give

$$(108) \quad x_1^6 x_2^4 x_3^7 = \frac{\rho \Delta_5 x_{12}}{\Delta},$$

where Δ and Δ_i ($i = 1, 2, \dots, 10$) are the determinants that are functions of the ten constant coefficients and of

X_i ($i = 1, 2, 3, 4, 5, 6, 8, 9, 10, 11$). Now from T' one obtains

$$(109) \quad x_3^{17} = \rho X_{11}.$$

This gives with (108) the following

$$(110) \quad \frac{x_1^6 x_2^4}{x_3^{10}} = \frac{\Delta_5 x_{12}}{\Delta X_{11}}.$$

In like manner the ten equations give

$$(111) \quad x_1^{12} x_3^5 = \frac{\rho \Delta_1 x_{12}}{\Delta},$$

and T' yields

$$(112) \quad x_1^{17} = \rho X_1.$$

Now combine the above two equations,

$$(113) \quad \frac{x_3^5}{x_1^5} = \frac{\Delta_1 x_{12}}{\Delta X_1}.$$

Next, combine (110) and (113) to get

$$(114) \quad \frac{x_2^4}{x_1^4} = \frac{\Delta_1^2 \Delta_5 x_{12}^3}{\Delta^3 x_1^2 x_{11}} .$$

The transformation T' yields

$$(115) \quad x_1^7 x_2^9 x_3^9 = \rho x_5 .$$

Also, the ten equations give

$$(116) \quad x_2^8 x_3^9 = \frac{\rho \Delta_{10} x_{12}}{\Delta} .$$

These two equations combine to give

$$(117) \quad \frac{x_1^7}{x_2^7} = \frac{\Delta x_5}{\Delta_{10} x_{12}} .$$

Now combine (114) and (117) to give

$$(118) \quad \frac{x_2}{x_1} = \frac{\Delta_1^4 \Delta_5^2 x_5 x_{12}^5}{\Delta^5 \Delta_{10} x_1^4 x_{11}^2} .$$

The transformation T' also yields

$$(119) \quad x_1 x_2^5 x_3^{11} = \rho x_9 ,$$

and the ten equations give

$$(120) \quad x_1 x_2^{13} x_3^3 = \frac{\rho \Delta_9 x_{12}}{\Delta} .$$

These two equations combine to give

$$(121) \quad \frac{x_3^8}{x_2^8} = \frac{\Delta X_9}{\Delta_9 X_{12}} .$$

From the transformation T^1 obtain

$$(122) \quad x_1^2 x_2^{10} x_3^5 = \rho X_8 .$$

The ten equations give

$$(123) \quad x_1^2 x_2 x_3^{14} = \frac{\rho \Delta_8 X_{12}}{\Delta} .$$

The above two equations combine to give

$$(124) \quad \frac{x_2^9}{x_3^9} = \frac{\Delta X_8}{\Delta_8 X_{12}} .$$

Now (121) and (124) combine to give

$$(125) \quad \frac{x_2}{x_3} = \frac{\Delta^2 X_8 X_9}{\Delta_8 \Delta_9 X_{12}^2} .$$

Finally combine (118) and (125) to give the inverse of the transformation T^1 . It is

$$(T^1)^{-1} \quad \frac{x_1}{\Delta^7 \Delta_{10} X_1^4 X_8 X_9 X_{11}^2} = \frac{x_2}{\Delta^2 \Delta_1^4 \Delta_5^2 X_5 X_8 X_9 X_{12}^5} \\ = \frac{x_3}{\Delta_1^4 \Delta_5^2 \Delta_8 \Delta_9 X_5 X_{12}^7} .$$

Hence, there is a one-to-one correspondence between the plane and the surface F , even though there is a one-to-seventeen correspondence between the surface ϕ and the plane and also between ϕ and F .

Conclusion

Using an homography, an involution of period seventeen was generated; and certain surfaces obtained from this involution were investigated. A family of plane curves invariant under this involution was projected, by means of the transformation T , to the hyperplanes of a space of ten dimensions (S_{10}). From this a surface ϕ , with points on it in a one-to-seventeen correspondence to the points of the plane, was arrived at. Then a study of the tangent elements at three branch points was carried out.

The next section of the study constituted a series of projections of this surface ϕ . The surfaces arrived at by successive projections were ϕ_i ($i = 1, 2, \dots, 7$). Then certain tangent elements at selected points on the various surfaces were exhibited.

By adding to the transformation T used previously, an additional coordinate X_{12} proportional to the function $g(x_1, x_2, x_3)$, a projectivity T' , mapping the points of the plane onto the points of a surface F in S_{11} , is established. Now each point of the plane is mapped onto a point on the surface F . By exhibiting the inverse of T' each point of F is mapped onto a point on the plane. Hence, the surface F is rational.

APPENDIX I

A METHOD OF FINDING THE ORDER OF A QUINTIC TANGENT CONE

There are various techniques of determining the order of a surface. The particular method illustrated here employs the definition from Woods [26, p. 390].

Examine as an example the surface (33). The equations of this surface combined with those of two general hyperplanes will give an homogeneous equation in two homogeneous variables. The degree of this final equation will be the order of the original surface.

Solve simultaneously the equations:

$$(126) \quad X_5 = X_7 = X_9 = X_{11} = 0$$

$$(127) \quad X_2^2 - X_1 X_3 = 0$$

$$(128) \quad X_3^2 - X_2 X_4 = 0$$

$$(129) \quad X_4^2 - X_3 X_6 = 0$$

$$(130) \quad X_6^2 - X_4 X_8 = 0$$

$$(131) \quad \sum A_i X_i = 0 \quad (i = 1, 2, \dots, 11)$$

$$(132) \quad \sum B_j X_j = 0 \quad (j = 1, 2, \dots, 11).$$

The combination of equations (126), (127), (131), and (132) will give after simplification

$$\begin{aligned}
 (133) \quad & (A_{110}B_{10} - A_{101}B_1) X_2^2 + (A_{210}B_{10} - A_{102}B_2) X_2X_3 + (A_{310}B_{10} - A_{103}B_3) X_3^2 \\
 & + (A_{410}B_{10} - A_{104}B_4) X_3X_4 + (A_{610}B_{10} - A_{106}B_6) X_3X_6 \\
 & + (A_{810}B_{10} - A_{108}B_8) X_3X_8 = 0.
 \end{aligned}$$

Now substitute equation (128) to eliminate X_2 and then use (129) to remove X_3 ,

$$\begin{aligned}
 (134) \quad & (A_{110}B_{10} - A_{101}B_1) X_4^4 + (A_{210}B_{10} - A_{102}B_2) X_4^3X_6 + (A_{310}B_{10} - A_{103}B_3) X_4^2X_6^2 \\
 & + (A_{410}B_{10} - A_{104}B_4) X_4X_6^3 + (A_{610}B_{10} - A_{106}B_6) X_6^4 \\
 & + (A_{810}B_{10} - A_{108}B_8) X_6^3X_8 = 0.
 \end{aligned}$$

Now employ (130) to arrive at

$$\begin{aligned}
 (135) \quad & (A_{110}B_{10} - A_{101}B_1) X_6^5 + (A_{210}B_{10} - A_{102}B_2) X_6^4X_8 + (A_{310}B_{10} - A_{103}B_3) X_6^3X_8^2 \\
 & + (A_{410}B_{10} - A_{104}B_4) X_6^2X_8^3 + (A_{610}B_{10} - A_{106}B_6) X_6X_8^4 \\
 & + (A_{810}B_{10} - A_{108}B_8) X_8^5 = 0.
 \end{aligned}$$

Note that the solution of the fifth degree equation (135) indicates that the tangent element is a quintic cone.

APPENDIX II

A METHOD OF INVESTIGATING A FOURTEENTH ORDER NEIGHBORHOOD

In the investigation of involutions that involve large values of p such that $E^p = 1$, there arises the problem of applying a quadratic transformation repetitively to a large equation. For example, in the study of O_1^1 , a quadratic transformation R had to be applied fourteen times to a seventeenth degree equation, cf., Chapter II.

The problem is not quite as difficult as the reader might first expect. The use of homogeneous coordinates makes the computation slightly less involved.

The following is a description of how a pattern develops. Observe that a term $z_1^i z_2^j z_3^k$ under R goes into $z_1^{2i+j} z_2^j + k z_3^k$. The a_{170} term stops any factoring of z_3 's in the simplification. Thus for any given term the k , or the exponent of z_3 , remains constant under applications of R . The a_9 term allows only one z_2 to be factored out in the simplification. Hence, for a given term the z_2 exponent will increase by the constant value $k - 1$ under each application. Now the a_{171} term allows only $z_1^{i+j+k-17}$ to be factored out. This final result after simplification is $z_1^{2i+j-i-j-k+17} z_2^{j+k-1} z_3^k$ or $z_1^{i-k+17} z_2^{j+k-1} z_3^k$. For a given term z_1 increases by the constant $17 - k$ and z_2 increases by the constant $k - 1$ for each application. This pattern develops only after one complete application of R .

The above explanation is not meant to be a proof that a similar constant increase pattern develops under certain types of quadratic transformations applied to any homogeneous equation, even though a related theorem might conceivably be constructed. The explanation is included here simply because it happened in all the applications of this paper and it was a considerable time saver.

The following chart of numbers is included as a display of the exponents of the z 's under fourteen applications of the transformation R to equation (18).

$a_9 x_1^{14} x_2^2 x_3$	$a_{26} x_1^{11} x_2^4 x_3^2$	$a_{52} x_1^8 x_2^6 x_3^3$	$a_{59} x_1^7 x_2 x_3^9$	$a_{87} x_1^5 x_2^8 x_3^4$
14 2 1	11 4 2	8 6 3	7 1 9	5 8 4
30 0 1	26 3 2	22 6 3	15 7 9	18 9 4
46 0 1	41 4 2	36 8 3	23 15 9	31 12 5
62 0 1	56 5 2	50 10 3	31 23 9	44 15 4
78 0 1	71 6 2	64 12 3	39 31 9	57 18 4
94 0 1	86 7 2	78 14 3	47 39 9	70 21 4
110 0 1	101 8 2	92 16 3	55 47 9	83 24 4
126 0 1	116 9 2	106 18 3	63 55 9	96 27 4
142 0 1	131 10 2	120 20 3	71 63 9	109 30 4
158 0 1	146 11 2	134 22 3	79 71 9	122 33 4
174 0 1	161 12 2	148 24 3	87 79 9	135 36 4
190 0 1	176 13 2	162 26 3	95 87 9	148 39 4
206 0 1	191 14 2	176 28 3	103 95 9	161 42 4
222 0 1	206 15 2	190 30 3	111 103 9	174 45 4
238 0 1	221 16 2	204 32 3	119 111 9	187 48 4

$a_{99} x_1^4 x_2^3 x_3^{10}$	$a_{133} x_1^2 x_2^{10} x_3^5$	$a_{148} x_1 x_2^5 x_3^{11}$	$a_{170} x_2^{17}$	$a_{171} x_3^{17}$
4 3 10	2 10 5	1 5 11	0 17 0	0 0 17
11 10 10	14 12 5	7 13 11	17 14 0	0 14 17
18 19 10	26 16 5	13 23 11	34 13 0	0 30 17
25 28 10	38 20 5	19 33 11	51 12 0	0 46 17
32 37 10	50 24 5	25 43 11	68 11 0	0 62 17
39 46 10	62 28 5	31 53 11	85 10 0	0 78 17
46 55 10	74 32 5	37 63 11	102 9 0	0 94 17
53 64 10	86 36 5	43 73 11	119 8 0	0 110 17
60 73 10	98 40 5	49 83 11	136 7 0	0 126 17
67 82 10	110 44 5	55 93 11	153 6 0	0 142 17
74 91 10	122 48 5	61 103 11	170 5 0	0 158 17
81 100 10	134 52 5	67 113 11	187 4 0	0 174 17
88 109 10	146 56 5	73 123 11	204 3 0	0 190 17
95 118 10	158 60 5	79 133 11	221 2 0	0 206 17
102 127 10	170 64 5	85 143 11	238 1 0	0 222 17

APPENDIX III

A METHOD OF DEMONSTRATING THE EXISTENCE OF $\Psi(X_1, X_2, \dots, X_{11})$

The following is one way of justifying that the terms of $(g(x_1, x_2, x_3))^{17}$ can be expressed in terms of the X_i 's ($i = 1, \dots, 11$) of the transformation T.

Suppose that the $a_{17}, a_{29}, \dots, a_{169}$ terms are raised to the powers b_1, b_2, \dots, b_{10} respectively and one factors out $\prod X_i^j$ where $i = 1, 2, \dots, 11$ and $j = d_1, d_2, \dots, d_{11}$. Then $(g(x_1, x_2, x_3))^{17}$ will be expressible as products of X_i 's if all possible combinations of integral values of $b_i = 0, 1, \dots, 17$, and $\sum b_j = 17$ are such that the equations (136), (137), and (138) have integral solutions of d_i 's where $\sum d_i = 17$. Equations (136), (137), and (138) are the conditions that cause the exponents of the x_1, x_2 , and x_3 to be the same in the powers and the factors.

$$(136) \quad 12 b_1 + 10 b_2 + 9 b_3 + 7 b_4 + 6 b_5 + 4 b_6 + 3 b_7 + 2 b_8 + b_9 \\ = 17 d_1 + 14 d_2 + 11 d_3 + 8 d_4 + 7 d_5 + 5 d_6 + 4 d_7 + 2 d_8 + d_9.$$

$$(137) \quad 7 b_2 + 2 b_3 + 9 b_4 + 4 b_5 + 11 b_6 + 6 b_7 + b_8 + 13 b_9 + 8 b_{10} \\ = 2 d_2 + 4 d_3 + 6 d_4 + d_5 + 8 d_6 + 3 d_7 + 10 d_8 + 5 d_9 + 17 d_{10}.$$

$$\begin{aligned}
 (138) \quad & 5 b_1 + 6 b_3 + b_4 + 7 b_5 + 2 b_6 + 8 b_7 + 14 b_8 + 3 b_9 + 9 b_{10} \\
 & = d_2 + 2 d_3 + 3 d_4 + 9 d_5 + 4 d_6 + 10 d_7 + 5 d_8 + 11 d_9 + 17 d_{11}.
 \end{aligned}$$

Now use $\sum b_i = 17$ and $\sum d_i = 17$ to eliminate b_8 and d_{11} in equations (136), (137), and (138). The results are

$$\begin{aligned}
 (139) \quad & 34 + 10 b_1 + 8 b_2 + 7 b_3 + 5 b_4 + 4 b_5 + 2 b_6 + b_7 - b_9 - 2 b_{10} \\
 & = 17 d_1 + 14 d_2 + 11 d_3 + 8 d_4 + 7 d_5 + 5 d_6 + 4 d_7 + 2 d_8 + d_9,
 \end{aligned}$$

$$\begin{aligned}
 (140) \quad & 17 - b_1 + 6 b_2 + b_3 + 8 b_4 + 3 b_5 + 10 b_6 + 5 b_7 + 12 b_9 + 7 b_{10} \\
 & = 2 d_2 + 4 d_3 + 6 d_4 + d_5 + 8 d_6 + 3 d_7 + 10 d_8 + 5 d_9 + 17 d_{10},
 \end{aligned}$$

$$\begin{aligned}
 (141) \quad & 51 + 9 b_1 + 14 b_2 + 8 b_3 + 13 b_4 + 7 b_5 + 12 b_6 + 6 b_7 + 11 b_9 \\
 & + 5 b_{10} = 17 d_1 + 16 d_2 + 15 d_3 + 14 d_4 + 8 d_5 + 13 d_6 + 7 d_7 \\
 & + 12 d_8 + 6 d_9 + 17 d_{10}.
 \end{aligned}$$

Observe that the above three equations are not independent, i.e., equations (139) and (140) added together give equation (141). Consequently, if (139) and (140) are satisfied then (141) will automatically be satisfied.

To exhibit that (139) and (140) have a common solution assume that the $b_2, b_3, b_4, b_5, b_6, b_7$ are each equated respectively to $d_4, d_5, d_6, d_7, d_8, d_9$. The problem now is reduced to showing that

equations (142) and (143) have a common solution.

$$(142) \quad 34 + 10 b_1 - b_9 - 2 b_{10} = 17 d_1 + 14 d_2 + 11 d_3,$$

$$(143) \quad 17 - b_1 + 12 b_9 + 7 b_{10} = 2 d_2 + 4 d_3 + 17 d_{10}.$$

Now eliminate b_9 in the above two equations to obtain

$$(144) \quad 25 + 7 b_1 - b_{10} = 12 d_1 + 10 d_2 + 8 d_3 + d_{10}.$$

The author has examined individually all the possible variations of b_1 and b_{10} in equation (144). They are considerably too many to be listed here.

The above analysis is included to explain to the reader how the number of test situations are substantially reduced.

BIBLIOGRAPHY

1. Childress, N. A. "Surfaces Obtained from Involutions Generated by Homographies of Period Three, Five, and Thirteen." Unpublished form of Ph.D. dissertation, Department of Mathematics, University of Florida, 1954.
2. Dessart, J. "Sur les surfaces representant l'involution engendree par une homographie de period cinq du plan," Memoires de la Societe Royale des Sciences de Liege, 3^e Series, Tome XVI (1931), pp. 1-23.
3. Frank, Stanley. "Certain Cyclic Involutory Mappings of Hyperspace Surfaces." Unpublished form of Ph.D. dissertation, Department of Mathematics, University of Florida, 1960.
4. Godeaux, Lucien. Cours de Geometrie Superieure. Fascicule II. Liege: Librairie Bourguignon, 1937.
5. _____. "Etude elementaire sur l'homographie plane de periode trois et sur une surface cubique," Nouvelles Annales de Mathematiques, 4^e Serie, Tome XVI (1916), pp. 49-61.
6. _____. Geometrie Algebrique. 2 tomes. Liege: Sciences et Lettres, 1948-1949.
7. _____. Introduction a la Geometrie Projective Hyper-spatiale. Liege: Librairie Bourguignon, 1939.
8. _____. Introduction a la Geometrie Superieure. 2^e edition. Liege: G. Thone, 1946.
9. _____. Memoire sur les Surfaces Multiples. Liege: G. Thone, 1952.
10. _____. "Recherches sur les involutions cycliques appartenant a une surface algebrique," Bulletin de l'Academie Royale de Belgique (Classes de Sciences), 5^e Series, Tome XIII (1931), pp. 1356-1364.
11. _____. "Sur les homographies plane cycliques," Memoires de la Societe Royale des Sciences de Liege, 3^e Serie, Tome XV (1930), pp. 1-26.
12. Gormsen, Svend T. "Maps of Certain Algebraic Curves Invariant under Cyclic Involutions of Period Three, Five and Seven." Unpublished form of Ph.D. dissertation, Department of Mathematics, University of Florida, 1953.

13. Graustein, William C. Introduction to Higher Geometry. New York: The Macmillan Company, 1946.
14. Hutcherson, W. F. "A Cyclic Involution of Order Seven," Bulletin of the American Mathematical Society, Vol. 40 (1934), pp. 143-151.
15. _____. "A Cyclic Involution of Period Eleven," Canadian Journal of Mathematics. Vol. III (1951), pp. 155-158.
16. _____. "Maps of Certain Cyclic Involutions on Two Dimensional Carriers," Bulletin of the American Mathematical Society, Vol. 37 (1931), pp. 759-765.
17. _____. "Invariant Curves of Order Eight," Revista Mathematica y Fisica Teorica, Serie A, Vol. 9 (1952), pp. 13-14.
18. _____, et Childress, N. A. "Etude d'une involution cyclique de periode cinq," Bulletin de l'Academie Royale de Belgique (Classes de Sciences), 6^e Serie, Tome XL (1954), pp. 103-106.
19. _____, and Childress, N. A. "Surfaces obtained from Involutions Generated by Homographies of Period Three, Five, and Thirteen," Revista Mathematica y Fisica Teorica, Serie A, Vol. 9 (1957), pp. 41-46.
20. _____, and Gormsen, S. T. "Maps of Certain Algebraic Curves Invariant Under Cyclic Involutions of Periods Three, Five, and Seven," Canadian Journal of Mathematics, Vol. VI (1954), pp. 92-98.
21. _____, and Kenelly, J. W., Jr. "An Involution of Period Seventeen Contained on a Rational Surface in a Space of 11 Dimensions," Notices of American Mathematical Society, Vol. 7 (1960), (Abstract) pp. 479.
22. _____, and Kenelly, J. W., Jr. "Three Branch Points on a Surface in a Space of Ten Dimensions," Notices of American Mathematical Society, Vol. 7 (1960), (Abstract) pp. 479-480.
23. Morelock, James C. "Invariants with Respect to Special Projective Transformations." Unpublished form of Ph.D. dissertation, Department of Mathematics, University of Florida, 1952.
24. Veblen, Oswald, and Young, John W. Projective Geometry. 2 vols. Boston: Ginn and Company, 1910-1918.

25. Winger, R. M. An Introduction to Projective Geometry.
Boston: D. C. Heath and Company, 1923.

26. Woods, Fredrick S. Higher Geometry. Boston: Ginn and
Company, 1922.

BIOGRAPHICAL SKETCH

John Willis Kenelly, Jr., was born at Bogalusa, Louisiana, on November 22, 1935. He attended the public schools of the City of Bogalusa and was graduated from Bogalusa High School in May, 1953. Immediately thereafter he entered Southeastern Louisiana College, Hammond, Louisiana, and completed the requirements for the degree of Bachelor of Science with a major in mathematics in August, 1956. This degree was conferred with honors at the following commencement exercises on May 25, 1957. He entered the University of Mississippi in the fall of 1956 and received the degree of Master of Science with a major in mathematics on August 18, 1957. Subsequently he entered the University of Florida in the fall of 1957 and has pursued graduate studies since that time, with the exception of the summer of 1960.

During his studies at the University of Florida the author has been a graduate assistant and later an instructor of mathematics. At the University of Mississippi he was a graduate fellow during the school year 1956-57 and a visiting Associate Professor for the summer term of 1960.

The author is an active member of the Mathematical Association of America and the American Mathematical Society.

The author married Charmaine Ruth Voss of Covington, Louisiana, in 1956. She is also a graduate of Southeastern Louisiana College.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

January 28, 1961

AH Groff
Dean, College of Arts and Sciences

Dean, Graduate School

SUPERVISORY COMMITTEE:

W. P. Stincherson
Chairman

J. E. May Jr.

W. P. House

J. O. Moore

G. R. Bartlett

